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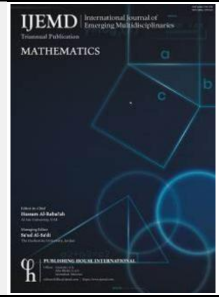

Homotopy Analysis Method Using Jumarie's Approach for Nonlinear Wave-like Equations of Fractional-order

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Abstract: In recent years, the study of fractional-order differential equations has gained significant attention due to its relevance in modeling various real-world phenomena with memory effects. Nonlinear wave equations of fractional order pose unique challenges due to the complex interplay of nonlinearity and fractional differentiation. This paper presents the application of the Homotopy Analysis Method (HAM) utilizing the modified Riemann–Liouville fractional derivative proposed by G. Jumarie to address these challenges. The proposed approach is designed to effectively tackle the complexities inherent in such equations, yielding highly promising results. Through a combination of numerical analyses and graphical representations, the algorithm's reliability and efficiency are comprehensively demonstrated.

Keywords: Modified Riemann–Liouvill; derivative, fractional differential equations; Homotopy Analysis, Wave-like Equation.

1. Introduction

In the last century, substantial advancements have been achieved in both the theoretical framework and practical applications of fractional differential equations. These equations have progressively gained traction within the realms of applied and engineering sciences [1-2]. Of particular significance are the wave-like equations, which hold pivotal roles across various domains including applied sciences,

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mathematical physics, nonlinear hydrodynamics, engineering physics, biophysics, human movement sciences, astrophysics, and plasma physics, as noted in [3] and the references cited therein. These equations serve as descriptors for the evolution of stochastic systems, exemplifying phenomena like the erratic motion of minuscule particles within fluid mediums, intensity fluctuations of laser light, velocity distribution of fluid particles in turbulent flows, and the stochastic tendencies observed in exchange rates. The resolution of nonlinear fractional differential equations encompasses a diverse array of techniques. Among these approaches are the utilization of the Mittag-Leffler function, the employment of the modified Riemann–Liouville derivative [4], the application of Homotopy analysis [5], the integration of Variational Iteration [6], the adoption of Differential Transform [7], the employment of Homotopy perturbation [8], the utilization of Modified Decomposition[9], the implementation of Reduced Differential Transform [10], the deployment of modified differential transformation [11], the exploitation of Exp-function [12], and the incorporation of both tanh [13] and Sine-cosine [14] functions. We consider the following nonlinear wave-like equations.

$$u_{tt} = \sum_{i,j=1}^{n} F_{1ij}(X,t,u) \frac{\partial^{k+m} F_{2ij}(u_{xi,} u_{xj})}{\partial x_i^{\,k} \partial x_i^{\,m}} + \sum_{i=1}^{n} G_{1i}\left(X,t,u\right) \frac{\partial^p}{\partial x^p} G_{2i}\left(u_{xi}\right) + H\left(X,t,u\right) + S\left(X,t\right),$$

with the initial conditions $u\left(X,0\right) = a_0(X), \dfrac{\partial u\left(X,0\right)}{\partial t} = a_1(X), \quad X = (x_1, x_2, x_3, \ldots, x_n)$ and F_{1ij}, G_{1i} are nonlinear functions of $X, t, u.$ F_{2ij}, G_{2i} are nonlinear functions of derivatives of x_i, x_j. while H, S are nonlinear functions and k, m, p are integers. In 1992, S.J. Liao have proposed a new technique (Homotopy Analysis Method) in their PhD thesis for solving linear and nonlinear equations [15 - 21] also worked on this method. Some initial and boundary-value problems are solved by using (HAM) [22-26]. The realm of fractional differential equations has evolved significantly over the past century, yielding profound impacts on theoretical insights and practical applications. Within this landscape, wave-like equations hold paramount importance in diverse scientific and engineering contexts. These equations encapsulate the behavior of stochastic systems, elucidating phenomena as varied as particle motion in fluids and the stochastic attributes of financial exchange rates. The toolbox for solving nonlinear fractional differential equations is rich and diverse, encompassing a spectrum of methodologies that draw from specialized mathematical functions and iterative procedures. This panoply of techniques underscores the breadth and depth of research in this field, offering avenues for comprehending and modeling complex phenomena that extend far beyond the traditional confines of differential calculus.

In the present article, we use Homotopy Analysis Method (HAM) using Jumarie's Approach for nonlinear Wave like Equations. Numerical results are highly encouraging.

2. Basic Definitions

We give some basic definitions and properties of the fractional calculus theory which are used

Further in this paper.

Definition: 2.1 G. Jumarie defined the fractional derivative [30] as the following limit

$$f^{\alpha} = \lim_{h\to 0} \frac{\Delta^{\alpha}[f(x)-f(0)]}{h}, \tag{2.1}$$

this definition is close to the standard definition of derivatives, and as a direct result, the αth Derivative of a constant, $0<\alpha<1$ is zero.

Definition: 2.2 Fractional integral operator of order $\alpha \geq 0$ is defined as

$$j_t^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\alpha} (x-s)^{\alpha-1} f(s) ds^{\alpha}, \alpha \geq 0. \tag{2.2}$$

Definition 2.3 The modified Riemann–Liouville derivative [33] is defined as

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha+1)} \frac{d^n}{dx^n} \int_0^{\alpha} (x-s)^{\alpha-1} (f(s)-f(0)) ds^{\alpha}, \alpha \geq 0. \tag{2.3}$$

Where $x \in [0,1]$ and $n-1 \leq \alpha \leq n, and n \geq 1.$

Definition 2.4 Fractional derivative of compounded functions [33] is defined as

$$d^{\alpha} f = \Gamma(\alpha+1) df, 0<\alpha<1. \tag{2.4}$$

Definition 2.5 The integral with respect to (dx) of [30] is defined as the solution of the fractional Differential equation

$$df = f(x) dx^{\alpha}, y(0)=0, 0<\alpha<1. \tag{2.5}$$

The definition of the fractional integral in Eq. (2.2) is equivalent to Lemma 4.1 of [30],

$$y=\int_0^x f(s)ds^{\alpha}=a\int_0^x (x-s)^{\alpha-1} f(s)ds^{\alpha}, \alpha\geq 0 . \quad (2.6)$$

For example, using $f(x)=x\gamma$ equation (2.6) one obtains,

$$\int_0^x (s)^{\alpha-1} f(s)ds^{\alpha}=\frac{\Gamma(\alpha+1)y(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}, 0<\alpha\leq 1. \quad (2.7)$$

3. Homotopy Analysis Method (HAM)

We consider the following equation

$$\tilde{N}\left[u(\tau)\right]=0, \quad (1.20)$$

where $\tilde{N}$ is a nonlinear operator, τ denotes dependent variables and $u(\tau)$ is an unknown function. For simplicity, we ignore all boundary and initial conditions, which can be treated in the similar way. By means of HAM Liao [1-7,47-48,65-73,113] constructed zero-order deformation equation

$$(1-p)\mathrm{L}\left[\phi(\tau;p)-u_0(\tau)\right]=p\hbar\tilde{N}\left[\phi(\tau;p)\right], \quad (1.21)$$

where L is a linear operator, $u_0(\tau)$ is an initial guess. $\hbar\neq 0$ is an auxiliary parameter and $p\in[0,1]$ is the embedding parameter. It is obvious that when p=0 and 1, it holds

$$\mathrm{L}\left[\phi(\tau;0)-u_0(\tau)\right]=0 \Rightarrow \phi(\tau;0)=u_0(\tau), \quad (1.22)$$

$$\hbar\tilde{N}\left[\phi(\tau;1)\right]=0\Rightarrow\phi(\tau;1)=u(\tau), \quad (1.23)$$

respectively. The solution $\phi(\tau;p)$ varies from initial guess $u_0(\tau)$ to solution $u(\tau)$. Liao expanded $\phi(\tau;p)$ in Taylor series about the embedding parameter

$$\phi(\tau; p) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau) p^m, \qquad (1.24)$$

where $$u_m(\tau) = \frac{1}{m!} \frac{\partial^m \phi(\tau; p)}{\partial p^m} \Big|_{p=0}. \qquad (1.25)$$

The convergence of (1.24) depends on the auxiliary parameter $\hbar$. If this series is convergent at $p = 1$,

$$\phi(\tau;1) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau). \qquad (1.26)$$

Define vector

$$\tilde{u}_n = \{u_0(\tau), u_1(\tau), u_2(\tau), u_3(\tau), \ldots\ldots\ldots, u_n(\tau)\}. \qquad (1.27)$$

If we differentiate the zeroth-order deformation equation eq. (1.21) m-times with respect to p and then divide them by $m!$ and finally set $p = 0$, we obtain the following mth-order deformation equation is given by

$$L\left[u_m(\tau) - X_m u_{m-1}(\tau)\right] = \hbar R_m(\tilde{u}_{m-1}), \qquad (1.28)$$

where

$$R_m\left(\tilde{u}_{m-1}\right) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \tilde{N}\left[\phi(\tau; p)\right]}{\partial p^{m-1}} \Big|_{p=0}, \qquad (1.29)$$

and

$$X_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1, \end{cases} \qquad (1.30)$$

we have Operating the Riemann–Liouville integral operator J^{α} on both side of Eq. (1.28), we have

$$u_m(\tau) = u^*_m(\tau) + X_m u_{m-1}(\tau) + J^{\alpha} \hbar R_m(\tilde{u}_{m-1}) \quad , \qquad (1.31)$$

where $u^*_m(\tau)$ is a particular solution, defined by

$$u^*_m(\tau) = j^{\alpha}\hbar \mathrm{R}_m(\tilde{u}_{m-1}) .$$

In this way, it is easily to obtain $u_1(\tau), u_2(\tau), u_3(\tau),..,$ one after another, at mth-order, we have

$$u(\tau) = \sum_{m=1}^{\infty} u_m(\tau).$$

When $m \to \infty$ we get an accurate approximation of the original equation.

4. Numerical Applications:

In this section, we apply the proposed algorithm to construct the solution of fractional PDEs. Numerical results obtained by the suggested scheme are encouraging. To check the efficiency, few examples are presented in this section.

Example 4.1: Two Dimensional Nonlinear Wave-like Equation

Consider the two dimensional non linear wave-like equation with variable coefficients

$$\frac{\partial^{2\alpha} u}{\partial \mathrm{t}^{2\alpha}} = \frac{\partial^2}{\partial x \partial y}\left(u_{xx}u_{yy}\right) - \frac{\partial^2}{\partial x \partial y}\left(xyu_x u_y\right) - u, \quad 0 < \alpha \le 1, \tag{4.1}$$

subject to the initial condition

$$u(x, y, 0) = e^{xy},$$

$$u_t(x, y, 0) = e^{xy}. \tag{4.2}$$

To solve equation (4.1) by HAM and using equation (2.2)

By taking $u_0(x, y, t) = e^{xy}(1+t)$, differentiating (1.21) with respect to q and setting $q = 0$, we have

$$u_1(x, y, t) = hL^{-1}\left[e^{xy}(1+t)\right],$$

$$u_1\left(x,y,t\right)=hJ^{2\alpha}\left[e^{xy}\left(1+t\right)\right],$$

$$u_1\left(x,y,t\right)=he^{xy}\left[\frac{t^{2\alpha}}{\Gamma\left(2\alpha+1\right)}+\frac{t^{2\alpha+1}}{\Gamma\left(2\alpha+2\right)}\right],$$

$$u_2\left(x,y,t\right)=h^2e^{xy}\left[\frac{t^{4\alpha}}{\Gamma\left(4\alpha+1\right)}+\frac{t^{4\alpha+1}}{\Gamma\left(4\alpha+2\right)}\right],$$

$$u_3\left(x,y,t\right)=h^3e^{xy}\left[\frac{t^{6\alpha}}{\Gamma\left(6\alpha+1\right)}+\frac{t^{6\alpha+1}}{\Gamma\left(6\alpha+2\right)}\right],$$

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The series solution is given by

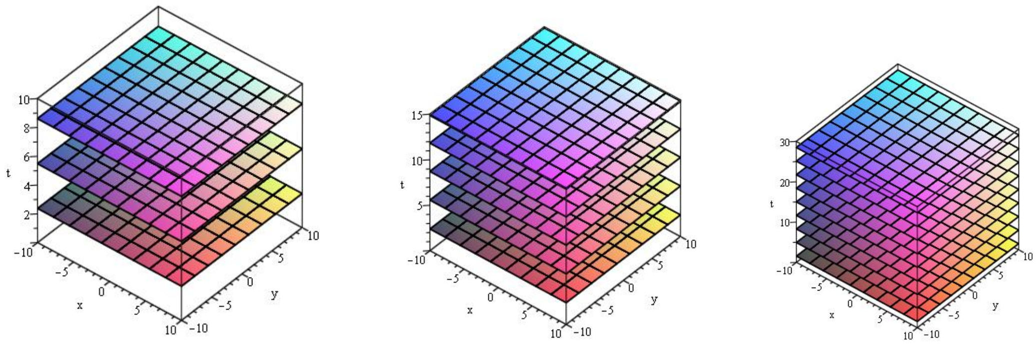
$$u\left(x,y,t\right)=u_0\left(x,y,t\right)+u_1\left(x,y,t\right)+u_2\left(x,y,t\right)+\ldots$$

$$u\left(x,y,t\right)=e^{xy}\left(1+t\right)+he^{xy}\left[\frac{t^{2\alpha}}{\Gamma\left(2\alpha+1\right)}+\frac{t^{2\alpha+1}}{\Gamma\left(2\alpha+2\right)}\right]+h^2e^{xy}\left[\frac{t^{4\alpha}}{\Gamma\left(4\alpha+1\right)}+\frac{t^{4\alpha+1}}{\Gamma\left(4\alpha+2\right)}\right]+\ldots$$ For

$\alpha=1 and\ h=-1$, we have

$$u\left(x,y,t\right)=e^{xy}\left[1+t-\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}-\ldots\right].$$

$$u\left(x,y,t\right)=e^{xy}\left[cost+sint\right].$$



$for\ -10 \le x \le 10,$ $-10 \le y \le 10, 0 \le t \le 10,$ (a)

$for\ -10 \le x \le 10,$ $-10 \le y \le 10, 0 \le t \le 15,$ (b)

$for\ -10 \le x \le 10,$ $-10 \le y \le 10, 0 \le t \le 30,$ (c)

Figure 4.1 (a)-(c) shows the behavior of $u(x, y, t)$ for different levels.

Example 4.2: One Dimensional Nonlinear Wave-like Equation with Exponential Conditions

Consider the nonlinear wave equation like with variable coefficients,

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} = u^2 \frac{\partial^2}{\partial x^2}\left(u_x u_{xx} u_{xxx}\right) + u^2{}_x \frac{\partial^2}{\partial x^2}\left(u^3{}_{xx}\right) - 18u^5 + u,\ \ 0 < \alpha \le 1,\ \ t > 0, \tag{4.3}$$

subject to the initial condition

$$u(x,0) = e^x,$$

$$u_t(x,0) = e^x. \tag{4.4}$$

To solve equation (4.3) by HAM and using equation (2.2), by taking

$u_0(x,t) = e^x(1+t)$, differentiating equation (1.21) with respect to q and setting $q = 0$, we have

$$u_1(x,t) = hL^{-1}\left[-e^x(1+t)\right],$$

$$u_1(x,y,t) = hJ^{2\alpha}\left[-e^x(1+t)\right],$$

$$u_1(x,y,t) = -he^x\left[\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right],$$

$$u_2(x,t) = h^2e^x\left[\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\right],$$

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The series solution is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots$$

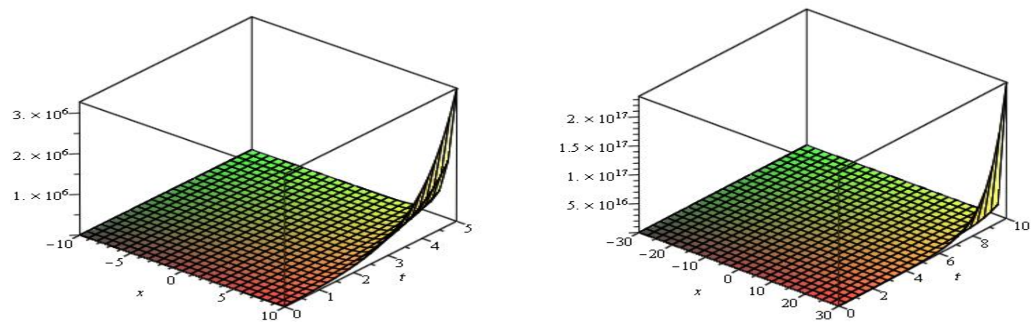
$$u(x,t) = e^x(1+t) - he^x\left[\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right] + h^2e^x\left[\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\right] + \ldots$$

For $\alpha = 1$ and $h = -1$, we have

$$u(x,t) = e^x\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \ldots\right].$$

$$u(x,t) = e^{x+t}.$$

Which is the exact solution.



(a) (b)

Fig 4.2 (a)-(b) shows the behavior of exact solution of $u(x,t)$ for (a) $-10 \le x \le 10, \quad 0 \le t \le 5,$ (b) $-30 \le x \le 30, \quad 0 \le t \le 10.$

Example 4.3: One-dimensional Nonlinear Wave-like Equation with Algebraic Conditions

Consider the nonlinear wave- like equation with variable coefficients,

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} = x^2 \frac{\partial}{\partial x}\left(u_x u_{xx}\right) - x^2\left(u^2{}_{xx}\right) - u, \quad 0 < \alpha \le 1, \quad t > 0, \quad 0 < x < 1, \tag{4.5}$$

subject to the initial conditions

$$u(x,0) = 0,$$

$$u_t(x,0) = x^2. \tag{4.6}$$

To solve equation (4.5) by HAM and using equation (2.2), by taking

$$u_0(x,t) = x^2 t,$$

Differentiating equation (1.21) with respect to q, and setting $q = 0$, we have

$$u_1(x,t) = hL^{-1}\left[x^2 t\right],$$

$$u_1(x,y,t) = hJ^{2\alpha}\left[x^2 t\right],$$

$$u_1(x,y,t) = hx^2\left[\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right],$$

$$u_1(x,y,t) = hx^2\left[\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right],$$

$$u_2(x,t) = h^2x^2\left[\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\right],$$

$$u_3(x,t) = h^3x^2\left[\frac{t^{6\alpha+1}}{\Gamma(6\alpha+2)}\right],$$

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The series solution is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \dots$$

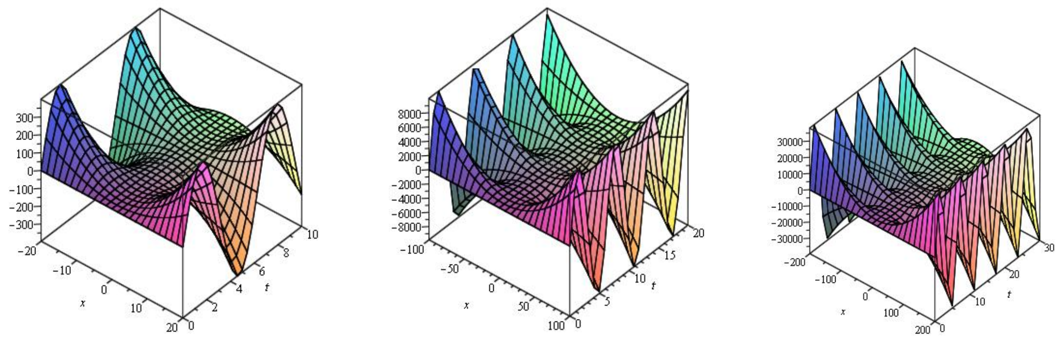
$$u(x,t) = x^2t + hx^2\left[\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right] + h^2x^2\left[\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\right] + h^3x^2\left[\frac{t^{6\alpha+1}}{\Gamma(6\alpha+2)}\right]\dots$$

For $\alpha = 1 and\ h = -1$, we have

$$u(x,t) = x^2\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots\right].$$

$$u(x,t) = x^2 sint.$$

Which is the exact solution.



(a) (b) (c)

Fig 4.3 (a)-(c) shows the behavior of exact solution of $u(x,t)$ for (a) $-20 \le x \le 20, 0 \le t \le 10,$ (b) $-100 \le x \le 100, 0 \le t \le 20,$ and (c) $-200 \le x \le 200, 0 \le t \le 30.$

5. Conclusions

In the present study, we employ the Homotopy Analysis Method (HAM) in conjunction with the modified Riemann–Liouville fractional derivative to address a set of nonlinear wave-like equations. Our investigation reveals that the proposed algorithm demonstrates an exceptional balance between simplicity and precision. Despite its straightforward implementation, it exhibits a remarkable degree of accuracy in resolving the intricate nonlinear wave-like equations under consideration. Furthermore, the versatility of the proposed approach is noteworthy, as it presents the potential for extension to a broader spectrum of both linear and nonlinear problems manifesting diverse physical characteristics. The results underscore the efficacy of the proposed methodology, positioning it as a valuable tool for addressing complex mathematical challenges in diverse scientific and engineering domains. This study thus establishes a foundation for future explorations into the application of the HAM via modified Riemann–Liouville fractional derivative across a wider array of problems, potentially offering novel insights and solutions to a range of intricate and multifaceted phenomena

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