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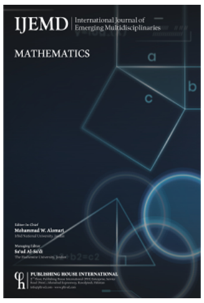

Homotopy Analysis Method for Fourth-Order Time Fractional Diffusion-Wave Equation

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Abstract

This paper is devoted to the study of a fourth-order fractional diffusion-wave equation defined in a bounded space domain. We apply Homotopy Analysis Method (HAM) to obtain solutions of fourth-order fractional diffusion-wave equation defined in a bounded space domain. It is observed that the HAM improves the accuracy and enlarge the convergence domain.

Keywords: Homotopy Analysis Method; fourth-order fractional diffusion-wave equation; bounded domain.

1. Introduction

Analysis of the diffusion-wave equation in mathematical physics have been of considerable interest in the literature. The time fractional diffusion-wave equation [1] is obtained from the classical diffusion or wave equation by replacing the first-or second order time derivative by a fractional derivative of order α with $0 < \alpha < 1$ or $1 < \alpha < 2$, respectively [2]. It is observed that as α increases from 0 to 2, the process changes from slow diffusion to classical diffusion to diffusion-wave to classical wave process. In the recent years several authors, for example, Mainardi [3, 4], Schneider and Wyss [5], and El-Sayed [6] have investigated the fractional diffusion/wave equation and its special properties. Fractional diffusion-wave equation has important applications to mathematical physics. Nigmatullin [7] has employed the fractional diffusion equation to describe diffusion in media with fractal geometry. Ginoa et al. [8] have presented a fractional diffusion equation describing relaxation phenomena in complex viscoelastic materials. In some applications, a fourth -order space derivative term is necessary. For example, wave propagation in beams and modeling formation of grooves on a flat surface because of grain require fourth-order space derivative terms in their formulations [9, 10, 11, 12, 13].

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Recently, Homotopy Analysis Method (HAM) [15-25] has been used for solving a wide range of problems, as it yields analytical solutions and offers certain advantages over standard numerical methods. This method is preferable over numerical methods as it is free from rounding off errors and neither requires large computer power/memory. O Bazighifan et al. [26, 27, 28, 29] studies the sufficient conditions for oscillation of all solutions of a second-order functional differential equation. Imran et al. [30, 31] obtain analytical solutions of nonlinear Schrödinger equations. Qayyum et al. [32] discuss thin film flow of non-Newtonian pseudo-plastic fluid is investigated on a vertical wall through homotopy-based scheme along with fractional calculus.

Agrawal [14] has solved fourth-order fractional-diffusion equation defined in a bounded space domain using finite sine transform technique. In the present article we employ the Homotopy Analysis Method for solving fourth-order fractional diffusion-wave equation. The paper has been organized as follows. Section 2 consists of analysis of Homotopy Analysis Method of the fourth-order fractional diffusion-wave equation has been developed. In Section 3 some illustrative examples are given. Discussion and conclusions are presented in Section 4.

Homotopy Analysis Method (HAM) [15-25]

We consider the following equation

$$\tilde{N}\left[u(\tau)\right]=0, \tag{1}$$

where $\tilde{N}$ is a nonlinear operator, τ denotes dependent variables and $u(\tau)$ is an unknown function. For simplicity, we ignore all boundary and initial conditions, which can be treated in the similar way. By means of HAM Liao [6-10] constructed zero-order deformation equation

$$(1-p)\mathrm{L}\left[\varnothing(\tau;p)-u_0(\tau)\right]=p\hbar\tilde{N}\left[\varnothing(\tau;p)\right], \tag{2}$$

where L is a linear operator, $u_0(\tau)$ is an initial guess. $\hbar\neq 0$ is an auxiliary parameter and $p\in[0,1]$ is the embedding parameter. It is obvious that when p=0 and 1, it holds

$$\mathrm{L}\left[\varnothing(\tau;0)-u_0(\tau)\right]=0 \Rightarrow \varnothing(\tau;0)=u_0(\tau), \tag{3}$$

$$\hbar\tilde{N}\left[\varnothing(\tau;1)\right]=0\Rightarrow\varnothing(\tau;1)=u(\tau), \tag{4}$$

respectively. The solution $\varnothing(\tau;p)$ varies from initial guess $u_0(\tau)$ to solution $u(\tau)$. Liao [18] expanded $\varnothing(\tau;p)$ in Taylor series about the embedding parameter

$$\varnothing(\tau;p)=u_0(\tau)+\sum_{m=1}^{\infty}u_m(\tau)p^m, \tag{5}$$

where

$$u_m(\tau)=\frac{1}{m!}\frac{\partial^m\varnothing(\tau;p)}{\partial p^m}\Big|_{p=0} \tag{6}$$

The convergence of (5) depends on the auxiliary parameter $\hbar$. If this series is convergent at p=1,

$$\varnothing(\tau;1)=u_0(\tau)+\sum_{m=1}^{\infty}u_m(\tau), \tag{7}$$

Define vector

$\vec{u}_n = \{u_0(\tau), u_1(\tau), u_2(\tau), u_3(\tau), \ldots\ldots\ldots, u_n(\tau)\}$

If we differentiate the zeroth-order deformation equation Eq. (2) m-times with respect to p and then divide them $m!$ and finally set $p=0$, we obtain the following m-th order deformation equation

$$\mathrm{L}\left[u_m(\tau) - X_m u_{m-1}(\tau)\right] = \hbar \mathrm{R}_m\left(\vec{u}_{m-1}\right), \tag{8}$$

where

$$\mathrm{R}_m\left(\vec{u}_{m-1}\right) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\tilde{N}\left[\varnothing(\tau;p)\right]}{\partial p^{m-1}}\Big|_{p=0} \tag{9}$$

and

$$X_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1, \end{cases} \tag{10}$$

If we multiply with L^{-1} each side of Eq. (8), we will obtain the following m-th order deformation equation

$$u_m(\tau) = X_m u_{m-1}(\tau) + \hbar \mathrm{R}_m\left(\vec{u}_{m-1}\right)$$

Numerical examples:

Example 1 Consider the following example of two dimensional time fractional fourth-order parabolic diffusion equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right), \qquad 0 < \alpha \le 1$$

with conditions

$$u(x,y,0) = e^{-x},$$

The linear operator is defined as

$$L[U(\tau;q)] = \frac{\partial^\alpha U}{\partial t^\alpha},$$

with the property that

$$L(C_0) = 0,$$

where C_0 is arbitrary constant.

If we choose $q \in [0,1]$ as embedding parameter, h as convergence control parameter then the zeroth-order deformation problem reads as

$$(1-q)L\left[U(\tau;q) - u_0(\tau)\right] = q\hbar N\left[U(\tau;q)\right],$$

and the nonlinear operator N is defined as

$$N\ \left[U(\tau;q)\right]=\frac{\partial U(\tau;q)}{\partial t}+\frac{\partial^4 U(\tau;q)}{\partial x^4},$$

Obviously, when $q=0$ and $q=1$, it holds

$$U(\tau,0)=u_0(\tau),\quad U(\tau,1)=u(\tau).$$

Thus as q increases from 0 to 1, $U(\tau;q)$ varies from the initial guess $u_0(\tau)$ to the final solution $u(\tau)$. We expand $U(\tau;q)$ in the Taylor series as

$$U(\tau,q)=u_0(\tau)+\sum_{m=1}^{\infty}u_m(\tau)q^m,\qquad u_m(\tau)=\frac{1}{m!}\frac{\partial^m U(\tau,q)}{\partial q^m}\bigg|_{q=0},$$

at $q=1$,

$$u(\tau)=u_0(\tau)+\sum_{m=1}^{\infty}u_m(\tau).$$

Differentiating m-times the zero-order deformation problem with respect to q and then setting $q=0$ and finally dividing by $m!$, we obtain the mth-order deformation problem given by

$$L[u_m(\tau)-\chi_m u_{m-1}(\tau)]=\hbar R_{1,m}(\tau),$$

Using the initial guess

$$u_0=e^{-x},$$

and using the m-th order deformations we have,

$$u_1=\hbar\frac{e^{-x}}{\Gamma(\alpha+1)}t^{\alpha},$$

$$u_2=\hbar^2\frac{e^{-x}}{\Gamma(2\alpha+1)}t^{2\alpha},$$

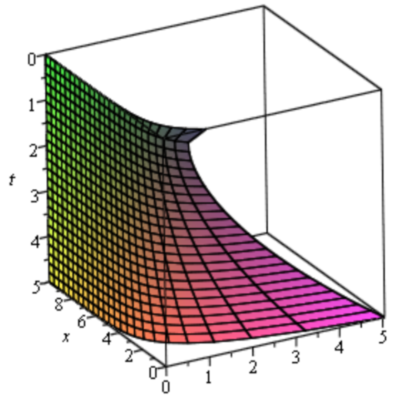
$$u_3=\hbar^3\frac{e^{-x}}{\Gamma(3\alpha+1)}t^{3\alpha},$$

and so on.

The closed form of solution is

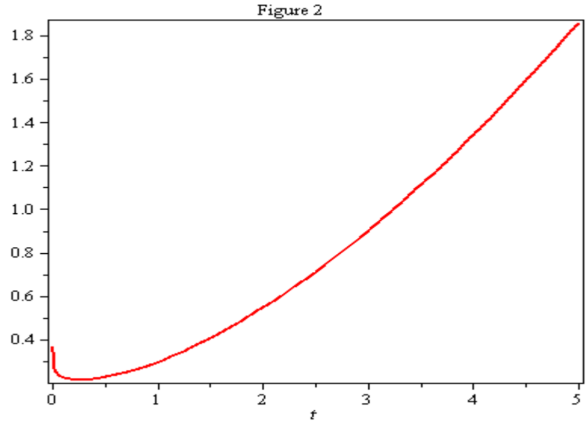
$$u=e^{-x}[1+\sum_{m=1}^{\infty}\frac{(\hbar t^{\alpha})^m}{\Gamma(m\alpha+1)}]$$

Figure 1



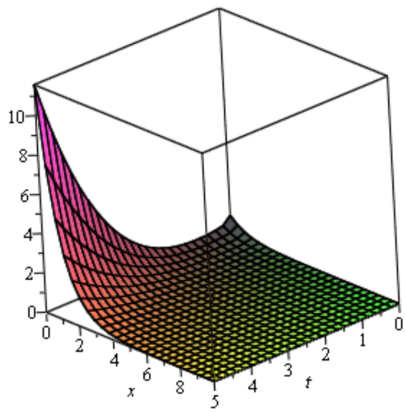
α = 0.25

Figure 2



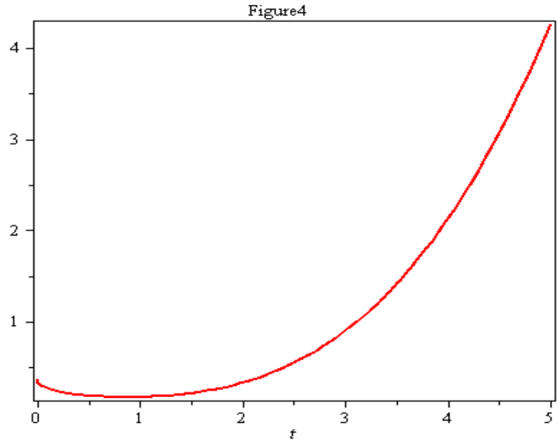
α = 0.25

Figure 3



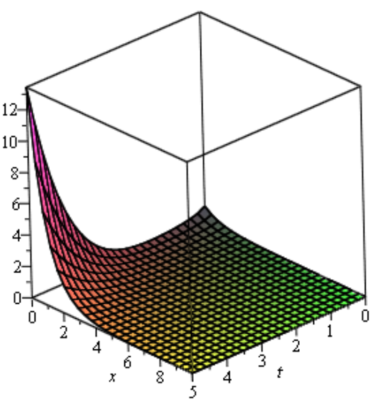
α = 0.5

Figure4



α = 0.5

Figure 5



α = 0.75

Figure 6

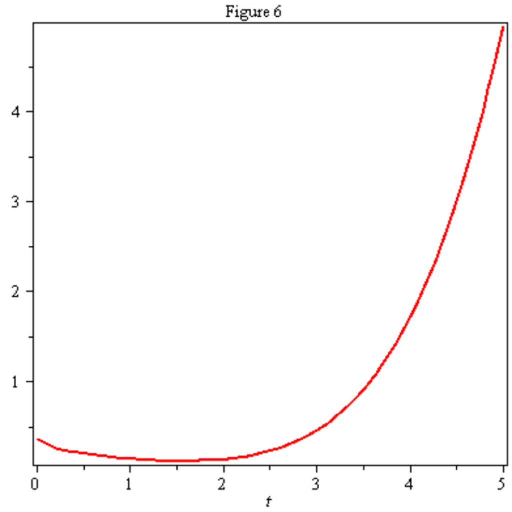
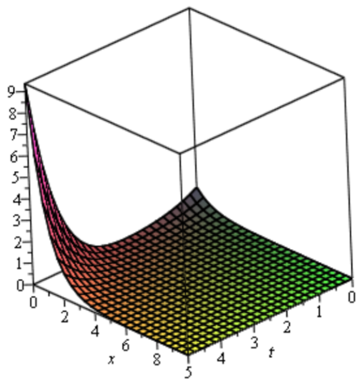


Figure 7



α = 1

Figure 8

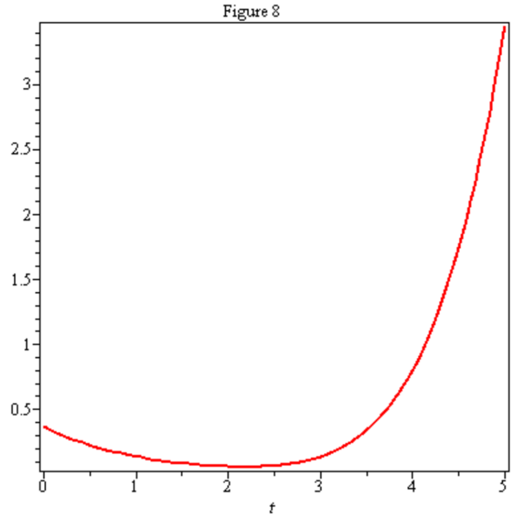


Table 1

t	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1.0$
0	e^{-1}	e^{-1}	e^{-1}	e^{-1}
0.5	$0.6180571596e^{-1}$	$0.5287949481e^{-1}$	$0.5537256922e^{-1}$	$0.6065321181e^{-1}$
1	$0.8066939209e^{-1}$	$0.4851336102e^{-1}$	$0.3973967989e^{-1}$	$0.3680555556e^{-1}$
1.5	$1.103072459e^{-1}$	$0.5943608461e^{-1}$	$0.3355955946e^{-1}$	$0.2259765625e^{-1}$
2	$1.482340678e^{-1}$	$0.907718320e^{-1}$	$0.3862749260e^{-1}$	$0.1555555559e^{-1}$

Example 2 Consider the following example of three dimensional time fractional fourth-order parabolic diffusion equation

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}} = -2\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} + \frac{\partial^4 u}{\partial z^4}\right), \qquad 0 < \alpha \le 1$$

with conditions

$$u(x, y, z, 0) = \cos(x)\cos(y)\cos(z),$$

The linear operator is defined as

$$L[U(\tau; q)] = \frac{\partial^{2\alpha}U}{\partial t^{2\alpha}},$$

with the property that

$$L(C_0 + C_1 t) = 0,$$

where C_0 *and* C_1 are arbitrary constants.

If we choose $q \in [0,1]$ as embedding parameter, $\hbar$ as convergence control parameter then the zeroth-order deformation problem reads as

$$(1-q)L\left[U(\tau;q)-u_0(\tau)\right]=q\hbar N\ \left[U(\tau;q)\right],$$

and the nonlinear operator N is defined as

$$N\ \left[U(\tau;q)\right]=\frac{\partial^2 U(\tau;q)}{\partial t^2}+2\left(\frac{\partial^4 U(\tau;q)}{\partial x^4}+\frac{\partial^4 U(\tau;q)}{\partial y^4}+\frac{\partial^4 U(\tau;q)}{\partial z^4}\right),$$

Obviously, when $q=0$ and $q=1$, it holds

$$U(\tau,0)=u_0(\tau),\quad U(\tau,1)=u(\tau).$$

Thus as q increases from 0 to 1, $U(\tau;q)$ varies from the initial guess $u_0(\tau)$ to the final solution $u(\tau)$. We expand $U(\tau;q)$ in the Taylor series as

$$U(\tau,q)=u_0(\tau)+\sum_{m=1}^{\infty}u_m(\tau)q^m,\qquad u_m(\tau)=\frac{1}{m!}\frac{\partial^m U(\tau,q)}{\partial q^m}\bigg|_{q=0},$$

at $q=1$,

$$u(\tau)=u_0(\tau)+\sum_{m=1}^{\infty}u_m(\tau).$$

Differentiating m-times the zero-order deformation problem with respect to q and then setting $q=0$ and finally dividing by $m!$, we obtain the mth-order deformation problem given by

$$L[u_m(\tau)-\chi_m u_{m-1}(\tau)]=\hbar R_{1,m}(\tau),$$

Using the initial guess

$$u_0=\cos(x)\cos(y)\cos(z)$$

and using the m-th order deformations we have,

$$u_1=\hbar\frac{4\cos(x)\cos(y)\cos(z)}{\Gamma(\alpha+1)}t^{\alpha},$$

$$u_2=\hbar^2\frac{16\cos(x)\cos(y)\cos(z)}{\Gamma(2\alpha+1)}t^{2\alpha},$$

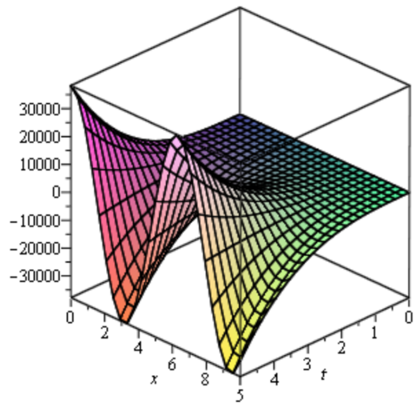
$$u_3=\hbar^3\frac{64\cos(x)\cos(y)\cos(z)}{\Gamma(3\alpha+1)}t^{3\alpha},$$

and so on.

The solution for n terms can be written as

$$u_n=\cos(x)\cos(y)\cos(z)\hbar^n 4^n\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}.$$

Figure 1



α = 0.25

Figure 2

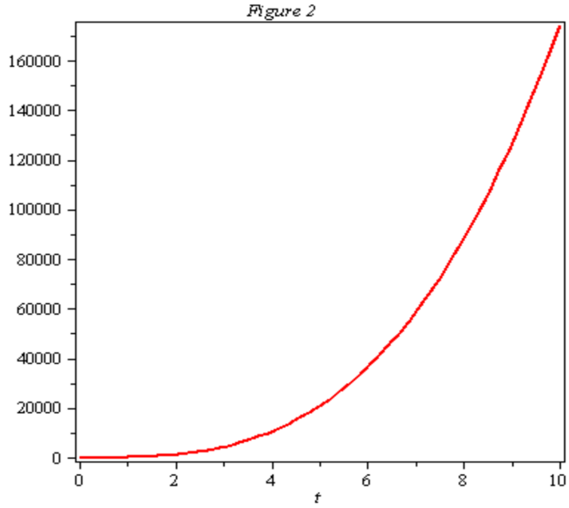
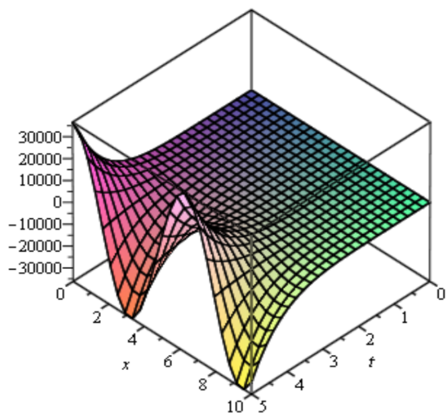


Figure 3



α = 0.5

Figure 4

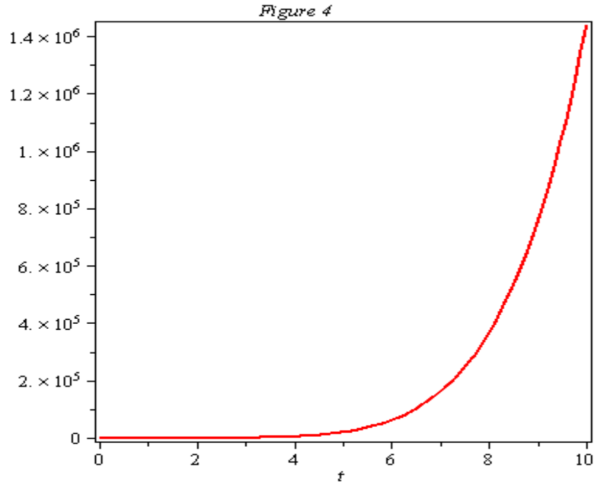
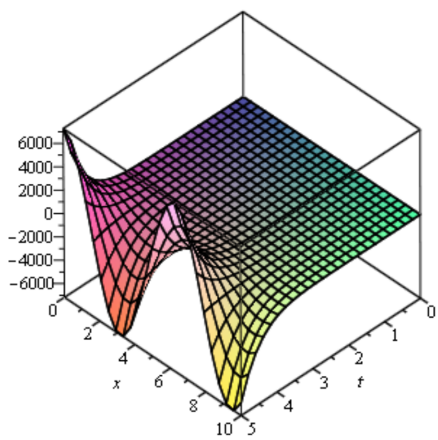
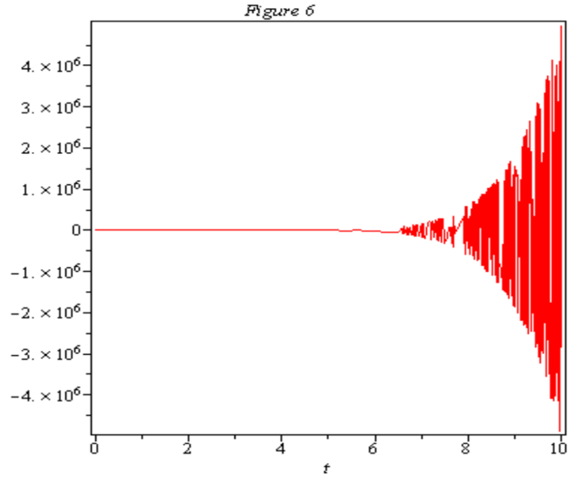


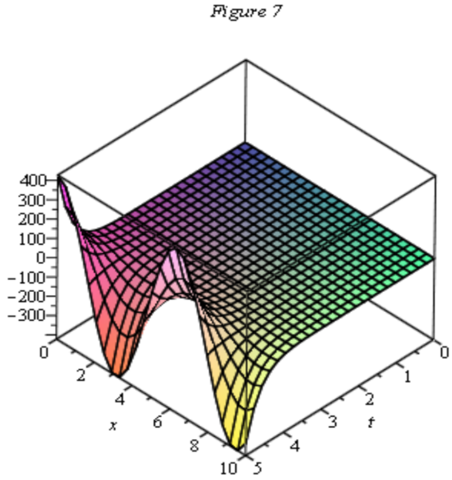
Figure 5



α = 0.75

Figure 6





$\alpha = 1$

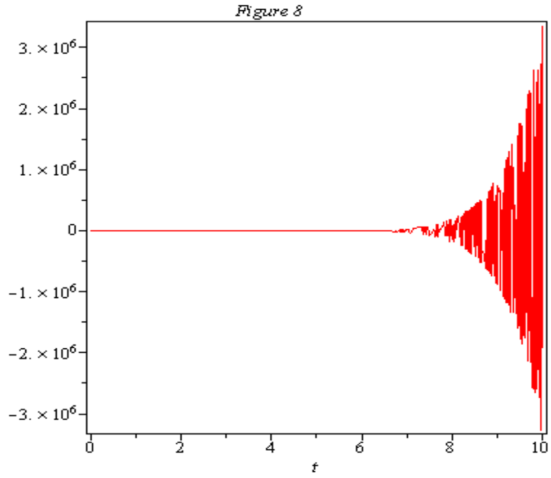


Table 2

t	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1.0$
0	0.2919265818	0.2919265818	0.2919265818	1
0.5	15.07839867	0.04541080153	0.06440634783	0.5403023059
1	136.2965025	0.629263966	–0.07916296510	–0.4161466517
1.5	488.6673406	9.049724044	–0.0443539458	–.9899396306
2	1202.963043	58.7810390	0.432852775	–0.6507594435

2. Conclusion

The Homotopy Analysis Method is a powerful tool which is capable of handling fractional partial differential equations. This method is better than numerical methods, as it is free from rounding off errors and does not require large computer power. The method has been successfully applied to fourth order fractional diffusion-wave equations. The computations associated with the illustrative examples in this paper were carried out using Maple.

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