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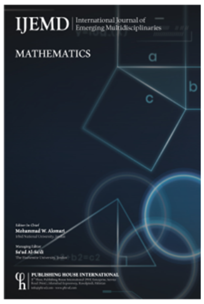

Homotopy Analysis Method for Free-Convective Boundary-Layer Equation Using Pade´Approximation

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Abstract

This paper is devoted to the study of a free-convective boundary layer flow modeled by a system of nonlinear ordinary differential equations. We apply Homotopy Analysis Method (HAM) along with Pade´ approximation to solve free-convective boundary-layer equation. It is observed that the combination of HAM and the Pade´ approximation improves the accuracy and enlarge the convergence domain.

Keywords: Homotopy Analysis Method; Pade´ techniques; similar boundary-layer; free-convective; nonlinear systems ordinary differential equations.

1. Introduction

Most of the physical phenomenon is nonlinear [1-20] in nature. The present manuscript reflects a comprehensive study on boundary-layer flows of viscous fluids [17-20] which are of utmost importance for industry and applied sceinces. These flows can be modeled by systems of nonlinear ordinary differential equations on an unbounded domain, see [19,21-25] and the refernces therein. Keeping in view the physical importance of such problems, there is a dire need of extension of some reliable and efficient technique for the solution of such problems. Liao [7-9, 20] developed the Homotopy Analysis Method (HAM) which is very efficient, accurate and is being used very frequently for finding the appropriate solutions of nonlinear problems of physical nature. The basic motivation of this paper is the application of Homotopy Analysis Method (HAM) coupled with Pade´ approximation to solve a free-convective boundary layer flow

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modeled by a system of nonlinear ordinary differential equations. Numerical and figurative illustrations show that it is a promising tool for solving nonlinear problems.

2. Homotopy Analysis Method (HAM) [1-20]

We consider the following equation

$$\tilde{N}\left[u(\tau)\right] = 0, \tag{1}$$

where $\tilde{N}$ is a nonlinear operator, τ denotes dependent variables and $u(\tau)$ is an unknown function. For simplicity, we ignore all boundary and initial conditions, which can be treated in the similar way. By means of HAM Liao [6-10] constructed zero-order deformation equation

$$(1-p)\mathrm{L}\left[\varnothing(\tau;p) - u_0(\tau)\right] = p\hbar\,\tilde{N}\left[\varnothing(\tau;p)\right], \tag{2}$$

where L is a linear operator, $u_0(\tau)$ is an initial guess. $\hbar \neq 0$ is an auxiliary parameter and $p \in [0,1]$ is the embedding parameter. It is obvious that when p=0 and 1, it holds

$$\mathrm{L}\left[\varnothing(\tau;0) - u_0(\tau)\right] = 0 \Rightarrow \varnothing(\tau;0) = u_0(\tau), \tag{3}$$

$$\hbar\,\tilde{N}\left[\varnothing(\tau;1)\right] = 0 \Rightarrow \varnothing(\tau;1) = u(\tau), \tag{4}$$

respectively. The solution $\varnothing(\tau;p)$ varies from initial guess $u_0(\tau)$ to solution $u(\tau)$. Liao [18] expanded $\varnothing(\tau;p)$ in Taylor series about the embedding parameter

$$\varnothing(\tau;p) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau)\,p^m, \tag{5}$$

where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \varnothing(\tau;p)}{\partial p^m}\Big|_{p=0} \tag{6}$$

The convergence of (5) depends on the auxiliary parameter $\hbar$. If this series is convergent at p=1,

$$\varnothing(\tau;1) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau), \tag{7}$$

Define vector

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), u_2(\tau), u_3(\tau), \ldots\ldots\ldots, u_n(\tau)\}$$

If we differentiate the zeroth-order deformation equation Eq. (2) m-times with respect to p and then divide them $m!$ and finally set $p = 0$, we obtain the following m-th order deformation equation

$$\mathrm{L}\left[u_m(\tau) - X_m u_{m-1}(\tau)\right] = \hbar \mathrm{R}_m\left(\vec{u}_{m-1}\right), \tag{8}$$

where

$$\mathrm{R}_m\left(\vec{u}_{m-1}\right) = \frac{1}{(m-1)!}\frac{\partial^{m-1} \tilde{N}\left[\varnothing(\tau;p)\right]}{\partial p^{m-1}}\Big|_{p=0} \tag{9}$$

and

$$X_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1, \end{cases} \tag{10}$$

If we multiply with L^{-1} each side of Eq. (8), we will obtain the following m-th order deformation equation

$$u_m(\tau) = X_m u_{m-1}(\tau) + \hbar \mathrm{R}_m\left(\vec{u}_{m-1}\right)$$

3. Mathematical Model

Let us consider the problem of cooling of a low-heat-resistance sheet that moves downwards in a viscous fluid

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{11}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2} + g\beta\left(T - T_0\right), \tag{12}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \kappa\frac{\partial^2 T}{\partial y^2}, \tag{13}$$

subject to

$$u = 0, \qquad v = 0 \;\text{ at }\; y=0, \tag{14}$$

$$u \to 0, \qquad \mathrm{T} \to \mathrm{T}_0 \text{ as } \mathrm{y} \to \infty, \tag{15}$$

where u and v are the velocity components in the x- and y- directions, respectively. T is the temperature , T_0 is the temperature of the surrounding fluid, v is the kinematic viscosity, κ is the thermal diffusivity, g is the acceleration due to gravity and β is the coefficient of thermal expansion. Using the similarity variables

$$\psi = \left[g\beta \left(T_1 - T_0\right) v^2 x_0^{\ 3} \right]^{1/4} f\left(\eta\right), \tag{16}$$

$$T = T_0 + \left(T_1 - T_0\right)\left[\frac{x_0}{\left(x_0 - x\right)}\right]^3 \theta\left(\eta\right), \tag{17}$$

$$\eta = \left[\frac{g\beta\left(T_1 - T_0\right) x_0^{\ 3}}{v^2}\right]^{1/4} \frac{y}{\left(x_0 - x\right)}, \tag{18}$$

where ψ is the stream function defined by $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, f and θ are the similarity functions dependent on η, $T\,(0,\,0) = T_1$ and $\theta(0) = 1$, (3.11)–(3.13) are transformed to

$$f'''(\eta) + \theta(\eta) - \left(f'(\eta)\right)^2 = 0, \tag{19}$$

$$\theta''(\eta) - 3\sigma f'(\eta)\theta(\eta) = 0, \tag{20}$$

subject to the boundary conditions

$$f(0) = 0, \qquad f'(0) = 0, \qquad f'(+\infty) = 0, \tag{21}$$

$$\theta(0) = 1, \qquad \theta(+\infty) = 0, \tag{22}$$

where the primes denote differentiation with respect to η and σ is the Prandtl number.

4. Pade´ Approximation

We denote L, M Pade´ approximants to $f\,(z)$ by

$$[L/M] = \frac{P_L(z)}{Q_M(z)} , \tag{23}$$

where $P_L(z)$ is polynomial of degree at most L and $Q_M(z)(Q_M(z) \neq 0)$ is a polynomial of degree at most M. The former power series is

$$f(z) = \sum_{k=0}^{\infty} c_k \cdot z^k , \tag{24}$$

And we write the $P_L(z)$ and $Q_M(z)$ as

$$P_L(z) = p_0 + p_1 \cdot z + p_2 \cdot z^2 + p_3 \cdot z^3 + \ldots + p_L \cdot z^L ,$$

$$Q_M(z) = q_0 + q_1 \cdot z + q_2 \cdot z^2 + q_3 \cdot z^3 + \ldots + q_M \cdot z^M , \tag{25}$$

so

$$f(z) - \frac{P_L(z)}{Q_M(z)} = O\left(z^{L+M+1}\right) \quad \text{as} \quad z \to 0 , \tag{26}$$

and the coefficients of $P_L(z)$ and $Q_M(z)$ are determined by the equation. From (4.4), we have

$$f(z) \cdot Q_M(z) - P_L(z) = O\left(z^{L+M+1}\right), \tag{27}$$

which system of $L+M+1$ homogeneous equations with $L+M+2$ unknown quantities. We impose the normalization condition

$$Q_M(0) = 1 . \tag{28}$$

We can write out (27) as

$$c_{L+1} + c_L \cdot q_1 + \ldots + c_{L-M+1} \cdot q_M = 0$$
$$c_{L+2} + c_{L+1} \cdot q_1 + \ldots + c_{L-M+2} \cdot q_M = 0 \qquad (4.7)$$
$$\vdots$$
$$c_{L+M} + c_{L+M-1} \cdot q_1 + \ldots + c_L \cdot q_M = 0,$$

$$c_0 = p_0$$
$$c_1 + c_0 \cdot q_1 = p_1$$
$$c_2 + c_1 \cdot q_1 + c_0 \cdot q_2 = p_2 \qquad (4.8)$$
$$\vdots$$
$$c_L + c_{L-1} \cdot q_1 + \ldots + c_0 \cdot q_L = p_L,$$

From (29) we can obtain the $q_i (1 \le i \le M)$.Once the values of $q_1, q_2, \ldots, q_M$ are all known (30) gives an explicit formula for the unknown quantities $p_1, p_2, \ldots, p_L$. Since the diagonal approximants like $[2/2], [3/3], [4/4], [5/5]\, or\, [6/6]$ have the most accurate approximants by built-in utilities of Maple.

5. Solution Procedure Consider the following problem formulated in section 2 and is related to the free-convective boundary layer flow

$$f'''(\eta) + \theta(\eta) - \left(f'(\eta)\right)^2 = 0,$$

$$\theta''(\eta) - 3\sigma f'(\eta)\theta(\eta) = 0,$$

subject to the boundary conditions

$$f(0) = 0, \quad f'(0) = 0, \quad f'(+\infty) = 0,$$

$$\theta(0) = 1, \quad \theta(+\infty)=0,$$

where the primes denote differentiation with respect to η and σ is the Prandtl number. The linear operator is defind as

$$L_1(f) = f''',$$

$$L_2(\theta) = \theta'',$$

with the property that

$$L_1(C_0 + C_1\eta + C_2\eta^2) = 0,$$

$L_2(C_3 + C_4\eta) = 0,$

where $C_i\,(i = 0-4)$ are arbitrary constants.

If we choose $q \in [0,1]$ as embedding parameter, h_f and h_θ as convergence control parameters then the zeroth-order deformation problem reads as

$$(1-q)L_1[F(\eta;q) - f_0(\eta)] = q\hbar_f N_1[F(\eta;q), \Theta(\eta;q)],$$

$$(1-q)L_2[\Theta(\eta;q) - \theta_0(\eta)] = q\hbar_\theta N_2[F(\eta;q), \Theta(\eta;q)],$$

and the nonlinear operators N_1 and N_2 are defined as

$$N_1[F(\eta;q), \Theta(\eta;q)] = \frac{\partial^3 F(\eta;q)}{\partial \eta^3} - \Theta(\eta;q) - \left(\frac{\partial F(\eta;q)}{\partial \eta}\right)^2,$$

$$N_2[F(\eta;q), \Theta(\eta;q)] = \frac{\partial^2 \Theta(\eta;q)}{\partial \eta^2} - 3\left(\frac{\partial F(\eta;q)}{\partial \eta}\Theta(\eta;q)\right),$$

Obviously, when $q = 0$ and $q = 1$, it holds

$$F(\eta,0) = f_0(\eta), \quad F(\eta,1) = f(\eta).$$

$$\Theta(\eta,0) = \theta_0(\eta), \quad \Theta(\eta,1) = \theta(\eta).$$

Thus as q increases from 0 to 1, $F(\eta;q)$ and $\Theta(\eta,\mathrm{q})$ varies from the initial guess $f_0(\eta)$ and $\theta_0(\eta)$ to the final solution $f(\eta)$ and $\theta(\eta)$. We expand $F(\eta;q)$ and $\Theta(\eta;q)$ in the Taylor series as

$$F(\eta,q) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)q^m, \qquad f_m(\eta) = \frac{1}{m!}\left.\frac{\partial^m F(\eta,q)}{\partial q^m}\right|_{q=0},$$

$$\Theta(\eta,q) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)q^m, \qquad \theta_m(\eta) = \frac{1}{m!}\left.\frac{\partial^m \Theta(\eta,q)}{\partial q^m}\right|_{q=0},$$

at $q = 1$,

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta).$$

$$\theta(\eta)=\theta_0(\eta)+\sum_{m=1}^{\infty}\theta_m(\eta).$$

Differentiating m -times the zero-order deformation problem with respect to q and then setting $q=0$ and finally dividing by $m!$, we obtain the mth -order deformation problem given by

$$L_1[f_m(\eta)-\chi_m f_{m-1}(\eta)]=\hbar R_{1,m}(\eta),$$

$$L_2[\theta_m(\eta)-\chi_m\theta_{m-1}(\eta)]=\hbar R_{2,m}(\eta),$$

Using the initial guess

$$f_0(\eta):=\frac{1}{2}\alpha_1\eta^2,\qquad \theta_0(\eta):=1+\alpha_2\eta,$$

and using the m-th order deformations we have,

$$f_1(\eta):=\frac{1}{6}\eta^3+\frac{1}{24}\alpha_2\eta^4-\frac{1}{60}\alpha_1{}^2\eta^5,$$

$$\theta_1(\eta)=-\frac{1}{4}\sigma\alpha_1\alpha_2\eta^4-\frac{1}{2}\sigma\alpha_1\eta^3,$$

$$f_2(\eta)=-\frac{1}{142560}\alpha_1{}^4\eta^{11}+\frac{1}{25920}\alpha_2\alpha_1{}^2\eta^{10}+\frac{1}{6048}\alpha_1{}^2\eta^9-\frac{1}{18144}\alpha_2{}^2\eta^9-\frac{1}{2016}\alpha_2\eta^8-\frac{1}{840}\sigma\alpha_1\alpha_2\eta^7$$
$$-\frac{1}{840}\eta^7-\frac{1}{240}\sigma\alpha_1\eta^6,$$

$$\theta_2(\eta)=-\frac{1}{1440}\sigma^2\alpha_1{}^3\alpha_2\eta^{10}+\frac{1}{576}\sigma^2\alpha_1\alpha_2{}^2\eta^9-\frac{1}{576}\sigma^2\alpha_1{}^3\eta^9+\frac{5}{488}\sigma^2\alpha_1\alpha_2\eta^8+\frac{1}{56}\sigma^2\alpha_1\eta^7,$$

and so on.

The series solution is

$$f(\eta)=\frac{1}{2}\alpha_1\eta^2+\frac{1}{6}\eta^3+\frac{1}{24}\alpha_2\eta^4-\frac{1}{60}\alpha_1{}^2\eta^5-\frac{1}{142560}\alpha_1{}^4\eta^{11}+\frac{1}{25920}\alpha_2\alpha_1{}^2\eta^{10}+\frac{1}{6048}\alpha_1{}^2\eta^9$$
$$-\frac{1}{18144}\alpha_2{}^2\eta^9-\frac{1}{2016}\alpha_2\eta^8-\frac{1}{840}\sigma\alpha_1\alpha_2\eta^7-\frac{1}{840}\eta^7-\frac{1}{240}\sigma\alpha_1\eta^6+\ldots.$$

$$\theta(\eta):=1+\alpha_2\eta-\frac{1}{4}\sigma\alpha_1\alpha_2\eta^4-\frac{1}{2}\sigma\alpha_1\eta^3-\frac{1}{1440}\sigma^2\alpha_1{}^3\alpha_2\eta^{10}+\frac{1}{576}\sigma^2\alpha_1\alpha_2{}^2\eta^9$$
$$-\frac{1}{576}\sigma^2\alpha_1{}^3\eta^9+\frac{5}{488}\sigma^2\alpha_1\alpha_2\eta^8+\frac{1}{56}\sigma^2\alpha_1\eta^7+\ldots$$

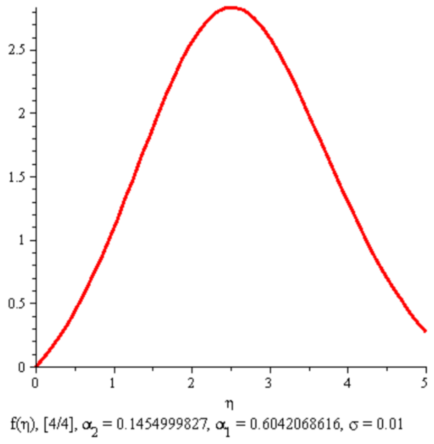

Figure 1

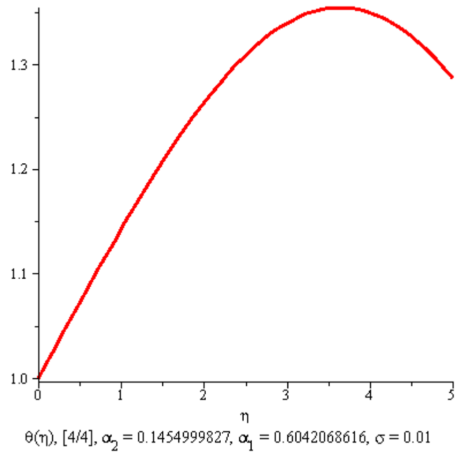

Figure 2

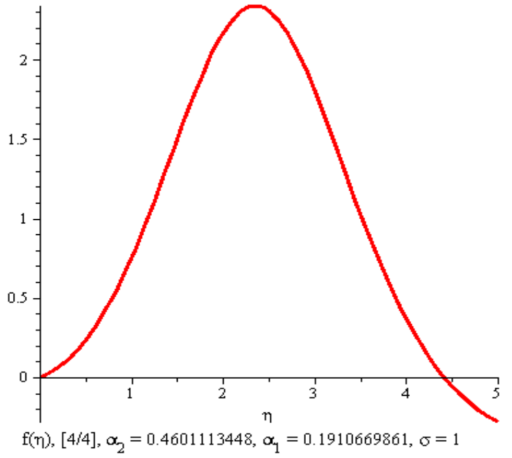

Figure 3

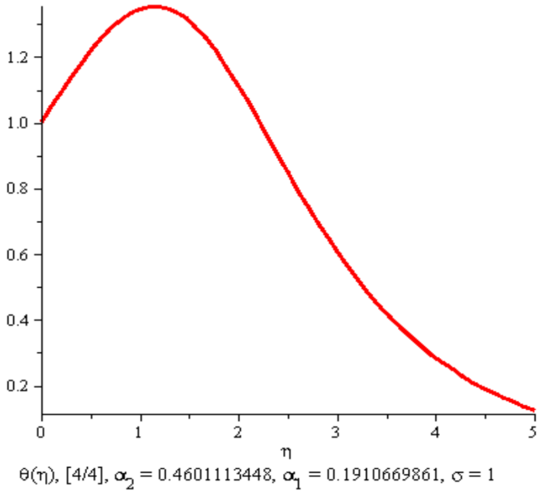

Figure 4

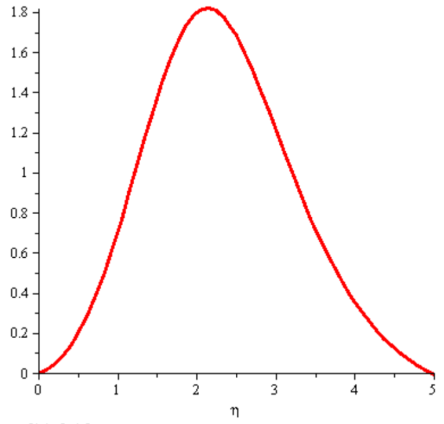

Figure 5

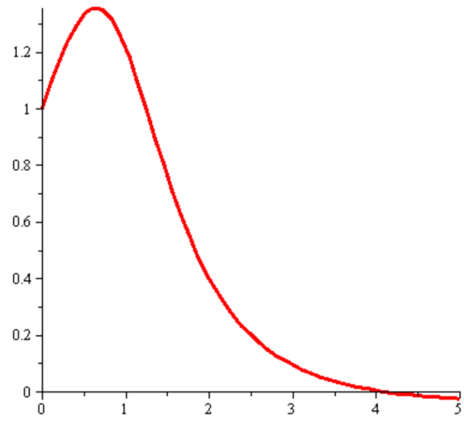

Figure 6

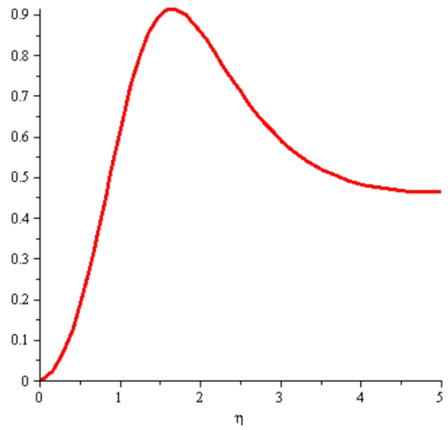

Figure 7

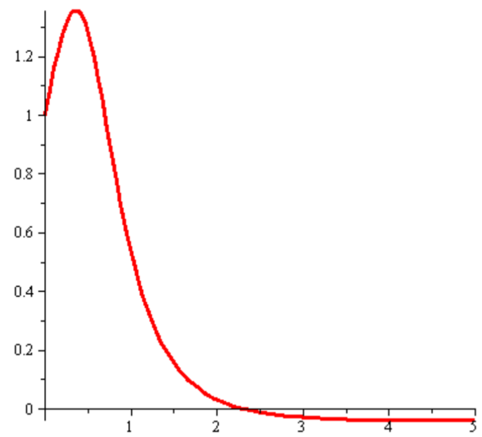

Figure 8

6. Conclusions

Homotopy Analysis Method (HAM) coupled with Pade´ approximation is employed to to solve a system of two nonlinear ordinary differential equations that describes a free-convective boundary layer in glass-fibre production process. The results show strong effects of the Prandtl number on the velocity and temperature profiles since the two model equations are coupled.

Data availability statement

The authors confirm that the data supporting the findings of this study are available within the article.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of the paper.

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