

Nonlinear Rosseland Radiation Effects on Unsteady MHD Mixed Convection of Casson Cu-CMC Nanofluid over an Inclined Stretching Sheet with Gravity Modulation

Noreen Niazi ¹, Tahir Naseem ^{2,3,*}, Hina Ayub ¹, Qurat-ul-Ain ¹, and Nagina Bibi ¹

1. Department of Mathematical, Government Girls Degree College No. 2 Haripur, 22620, Pakistan

2. Pak-Austria Fachhochschule Institute of Applied Sciences and Technology Mang, Haripur, 22600, Pakistan

3. Government Degree College Khanpur, Haripur, 22620, Pakistan, (ORCID ID: 0000-0003-1216-4128)

Abstract

A numerical investigation is conducted to analyze the unsteady mixed convective thermal behavior of a magnetized, radiatively active Casson-type Cu-CMC nanofluid moving over an angled stretching surface subjected to oscillatory gravitational effects. The study incorporates transverse magnetic forcing, Rosseland radiative diffusion, Joule dissipation, and viscous heating within the non-Newtonian nanofluid transport equations. Through appropriate similarity transformations, the governing partial differential system is reduced to a coupled set of nonlinear ordinary differential equations, which are subsequently solved using the implicit Keller-box finite-difference scheme. The nonlinear treatment of the Rosseland radiative flux, which retains the full temperature dependence of the absorption characteristics, modifies the effective thermal diffusion coefficient and consequently the radiation-modified Nusselt number. Systematic comparison with the published benchmark solutions for appropriate limiting cases confirms

the fidelity of the numerical implementation. Parametric investigation examines the sensitivity of the velocity and temperature fields, wall friction, and heat transfer rates to variations in the Casson rheological parameter, magnetic interaction coefficient, radiation parameter, Eckert number, surface inclination, nanoparticle loading, and gravity modulation characteristics. The results indicate that magnetic damping reduces the momentum boundary-layer thickness, copper nanoparticle addition enhances thermal penetration, and the combination of radiative transport with irreversible heating mechanisms substantially alters the thermal boundary-layer structure. These findings offer practical insights for the design and optimization of thermal management systems that depend on electrically conducting non-Newtonian nanofluids.

Keywords: Unsteady Casson nanofluid; gravity modulation; mixed convection; nonlinear thermal radiation; viscous dissipation.

1. Introduction

Boundary-layer flows over stretching surfaces constitute a fundamental paradigm in fluid mechanics, with direct applications across diverse industrial and manufacturing processes. Polymer extrusion, continuous casting of metals, glass fiber drawing, paper production, and thermal treatment of materials all involve the transport of fluids adjacent to moving solid boundaries. In these processes, the working medium frequently exhibits non-Newtonian behavior, and the transport characteristics are governed by the intricate interplay between mechanical stretching, thermal buoyancy, and external force fields. The Casson fluid model has emerged as a particularly useful rheological framework for describing the shear-thinning behavior with yield stress exhibited by many complex fluids, including blood, printing inks, polymer solutions, and concentrated colloidal suspensions [1-3]. Subsequent investigations have demonstrated that Casson rheology significantly influences the wall friction coefficient, thermal boundary-layer thickness, and convective heat transfer rates in both planar stretching and wedge-flow configurations [4-6].

The introduction of nanoparticles into conventional heat transfer fluids represents one of the most significant advancements in thermal engineering over the past three decades. The seminal work of Choi [7] established the nanofluid concept, demonstrating that dispersing nanoscale solid particles with high thermal conductivity can substantially enhance the effective thermal properties of the base fluid. Subsequent research efforts have systematically documented the influence of particle concentration on key transport parameters, including viscosity, density, specific heat capacity, thermal conductivity, and

thermal expansion coefficient [8-10]. Within the context of Casson nanofluid systems, these property modifications fundamentally alter both the momentum and thermal fields, particularly when the transport is further complicated by thermal radiation, internal heat generation, or reactive species [11-13].

Carboxymethyl cellulose (CMC)-water mixtures have attracted considerable attention as non-Newtonian base fluids due to their well-characterized shear-thinning behavior, stability, and biocompatibility. When combined with highly conductive copper nanoparticles, the resulting Cu-CMC nanofluid exhibits enhanced thermal transport capabilities while retaining the favorable rheological characteristics of the polymer solution. These properties make Cu-CMC nanofluids attractive candidates for cooling microelectronic devices, thermal management in heat exchangers, coating processes, and biomedical applications [14-16]. The unsteady nature of many practical thermal systems further motivates the consideration of time-dependent effects, particularly the periodic modulation of gravitational acceleration, which can significantly influence buoyancy-driven transport through parametric resonance and convective instability mechanisms [17]. Accurate modeling of the thermophysical properties, incorporating the contributions of the dispersed nanoparticles, remains essential for reliable quantitative predictions in these configurations [18].

Magnetohydrodynamic (MHD) effects provide a versatile means of controlling the motion of electrically conducting fluids through the action of Lorentz forces. A transverse magnetic field generates a resistive force opposing the flow, thereby modifying the velocity distribution and the associated wall shear stress. Simultaneously, the electromagnetic interaction can produce appreciable Joule heating, introducing an additional thermal source term in the energy equation. The influence of magnetic fields on non-Newtonian and nanofluid flows over stretching or porous surfaces has been extensively investigated in the literature [19, 20]. Classical boundary-layer studies have further established that nanoparticle loading, thermal radiation, and slip effects can substantially alter the heat transfer characteristics in MHD nanofluid systems [21-23]. For high-temperature environments and optically thick media, the Rosseland diffusion approximation provides a computationally tractable and physically sound approach for modeling radiative heat flux [24-26]. Investigations of radiative Casson fluid flows have consistently demonstrated that thermal radiation can profoundly modify the temperature field, particularly at elevated wall temperatures [27-29].

Recent research has extended these investigations to encompass more sophisticated transport phenomena, including entropy generation analysis, particle-shape effects, thin-film heat transfer, and hybrid

nanoparticle combinations. Naseem and Shahzad [12] examined radiative nanofluid transport with irreversibility effects and demonstrated that nanoparticle morphology significantly influences thermal transport characteristics. In a subsequent thin-film study, the same authors [30] investigated nanofluid heat transfer over an unsteady stretching surface, highlighting the role of entropy-related mechanisms in determining the overall system performance. Other contemporary works on copper-based nanofluids, axisymmetric liquid films, radiative power-law fluids, and modified hybrid nanofluids have reinforced the conclusion that magnetic field effects, radiation, and nanoparticle interactions must be carefully modeled to achieve reliable thermal predictions in practical systems [31, 32]. Investigations of hybrid nanofluid shape factors and porous or channel flows further support the development of integrated transport models capable of capturing the full complexity of real-world thermal management systems [33, 34].

While gravity-modulated, inclined stretching-sheet flow of Casson Cu-CMC nanofluids has been examined individually for select parameters, while individual aspects of this problem have been examined separately, the simultaneous consideration of time-periodic gravity modulation, full (nonlinearized) Rosseland radiation, Joule heating, and viscous dissipation in a Casson Cu-CMC nanofluid over an inclined stretching surface has not been reported. The combined effect of these mechanisms on the radiation-corrected Nusselt number is the principal novel contribution of this work. Critically, most existing treatments linearize the radiative flux about the ambient temperature — an approximation that breaks down under the steep wall-to-ambient temperature gradients typical of high-flux thermal management systems. The present work retains the full temperature-dependent Rosseland flux via the wall-temperature-ratio parameter θ_w , which introduces a genuinely nonlinear diffusion coefficient into the energy equation rather than a constant one. This is not a parameter addition on top of Rawi et al. [1]; it changes the mathematical character of the governing PDE and requires the Keller-box scheme's coefficients to be re-derived at each Newton iteration (Section 3.3), which is the paper's core numerical contribution.

The present study addresses this gap by developing a comprehensive magneto-radiative model for unsteady mixed convection of a Casson Cu-CMC nanofluid over an inclined stretching surface subjected to time-periodic gravity modulation. The formulation integrates the Lorentz force, Joule heating, viscous dissipation, and Rosseland radiation effects alongside the effective thermophysical properties of the nanoparticle suspension. The transformed nonlinear boundary-value problem is solved numerically using the implicit Keller-box finite-difference method, which offers robust stability and second-order accuracy.

The primary objectives of the investigation are to quantify the influence of the Casson parameter β , magnetic parameter M , radiation parameter Rd , Eckert number Ec , inclination angle α , nanoparticle volume fraction ϕ , and gravity modulation parameters on the velocity and temperature distributions, wall shear stress, and radiation-corrected Nusselt number. The integrated approach provides new physical insights that are directly relevant to thermal control systems employing electrically conducting non-Newtonian nanofluids under realistic operating conditions.

2. Physical Model and Assumptions

We study an unsteady, two-dimensional boundary-layer flow next to an impermeable sheet that stretches while tilted at angle α from vertical, as sketched in figure 1. Its surface moves with a linearly increasing stretching velocity $u_w(x) = cx$ and carries a wall temperature $T_w(x) = T_\infty + bx$, with c and b denoting positive constants. Copper nanoparticles are suspended in a CMC-water base fluid, which is treated as a Casson-type fluid possessing yield stress. Gravity is allowed to vary in time, modeled as a harmonic oscillation about a mean value g_0 with an oscillation amplitude of ϵ and a frequency of ω , in line with the g-jitter approach used by Sharidan et al. [17].

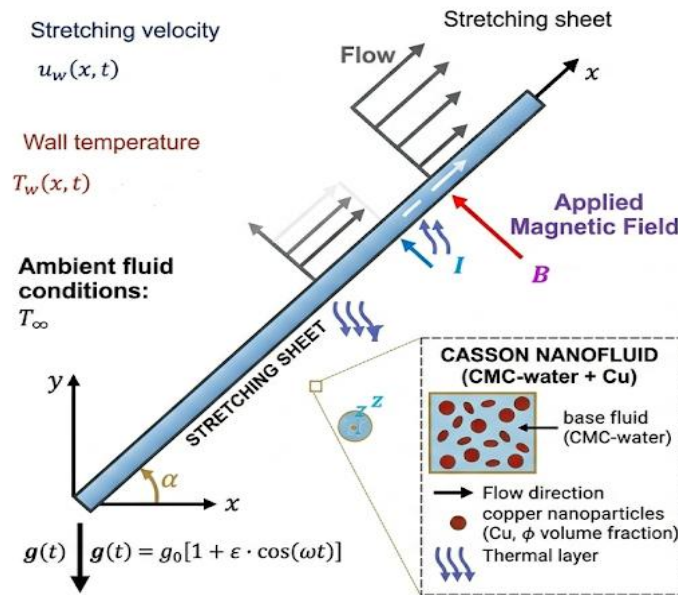


Fig.1. Physical Model and Coordinate System.

A uniform magnetic field of strength B_0 acts normal to the sheet. Since the magnetic Reynolds number is small, the induced electric field and Hall currents are dropped from the analysis. Motion in the boundary layer is driven jointly by the stretching of the sheet, buoyancy from the temperature difference, and the periodic gravity variation described above. Adapting the Casson-nanofluid modeling approach of Rawi et al. [1] to these assumptions, the governing mass, momentum, and energy balances read:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

Balancing momentum along the x-direction, including the Casson rheology, the Lorentz force, and buoyancy, gives:

$$\rho_{nf} \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \mu_{nf} \left(1 + \frac{1}{\beta} \right) \frac{\partial^2 u}{\partial y^2} + (\rho \beta_T)_{nf} g^*(t) (T - T_\infty) \cos \alpha - \sigma_{nf} B_0^2 u, \quad (2)$$

The present analysis considers the buoyancy-assisted configuration ($T_w > T_\infty$), where thermal expansion enhances the stretching-driven flow.

The corresponding energy balance takes the form:

$$(\rho C_p)_{nf} \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k_{nf} \frac{\partial^2 T}{\partial y^2} + \mu_{nf} \left(\frac{\partial u}{\partial y} \right)^2 + \sigma_{nf} B_0^2 u^2 - \frac{\partial q_r}{\partial y}. \quad (3)$$

The corresponding boundary conditions, consistent with [1,3], are

$$u = u_w(x), \quad v = 0, \quad T = T_w(x) \quad \text{at} \quad y = 0, \quad (4)$$

$$u \rightarrow 0, \quad T \rightarrow T_\infty \quad \text{as} \quad y \rightarrow \infty. \quad (5)$$

Standard mixture rules are used to evaluate the nanofluid's effective thermophysical properties. Density is taken as a volume-weighted average of the two constituents, and the heat capacity and thermal expansion coefficient are built the same way, as linear combinations of the base-fluid and nanoparticle values. Brinkman's correlation gives the dynamic viscosity, while the thermal conductivity follows from the Maxwell-Garnett effective-medium approximation. These relations are collected in the following expressions given in table 1:

Table 1. Thermal and Physical Parameters of the Nanofluid, following the effective-property correlations of Hayat et al. [10] and Ullah et al. [11].	
Property	Formula
Density	$\rho_{nf} = (1 - \phi)\rho_f + \phi\rho_s,$
Heat capacity	$(\rho C_p)_{nf} = (1 - \phi)(\rho C_p)_f + \phi(\rho C_p)_s,$
Thermal expansion	$(\rho\beta_T)_{nf} = (1 - \phi)(\rho\beta_T)_f + \phi(\rho\beta_T)_s,$
Dynamic viscosity	$\frac{\mu_{nf}}{\mu_f} = \frac{1}{(1 - \phi)^{2.5}},$
Thermal Conductivity	$\frac{k_{nf}}{k_f} = \frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + \phi(k_f - k_s)}.$
Electrical Conductivity	$\frac{\sigma_{nf}}{\sigma_f} = \frac{[\sigma_s + 2\sigma_f - 2\phi(\sigma_f - \sigma_s)]}{[\sigma_s + 2\sigma_f + \phi(\sigma_f - \sigma_s)]}$

The dimensionless material constants are denoted by:

$$C_1 = (1 - \phi) + \phi \frac{\rho_s}{\rho_f}, C_2 = (1 - \phi) + \phi \frac{(\rho\beta_T)_s}{(\rho\beta_T)_f}, C_3 = (1 - \phi) + \phi \frac{(\rho C_p)_s}{(\rho C_p)_f}.$$

Practical thermal-radiation systems often involve steep, non-uniform temperature gradients that push the linearized Rosseland approximation beyond its range of validity. To keep the thermal-transport model accurate under these conditions, we retain the full, non-linearized Rosseland radiative heat flux:

$$q_r = -\frac{4\sigma^*}{3k^*} \frac{\partial T^4}{\partial y},$$

where σ^* is the Stefan-Boltzmann constant and k^* is the mean absorption coefficient. A wall-temperature ratio θ_w , defined as:

$$\theta_w = \frac{T_w}{T_\infty}$$

As a result, the local temperature T within the boundary layer follows from the dimensionless temperature variable θ through the exact relation:

$$T = T_\infty[1 + (\theta_w - 1)\theta]$$

Putting this relation back into the Rosseland expression gives the non-linear radiative heat flux:

$$q_r = -\frac{16\sigma^*T_\infty^3}{3k^*}[1 + (\theta_w - 1)\theta]^3 \frac{\partial T}{\partial y}.$$

Differentiating q_r with respect to y gives the resulting radiative term in the energy equation:

$$-\frac{\partial q_r}{\partial y} = \frac{16\sigma^*T_\infty^3}{3k^*} \frac{\partial}{\partial y} \left\{ [1 + (\theta_w - 1)\theta]^3 \frac{\partial T}{\partial y} \right\}.$$

Following the transformation scheme of Rawi et al. [1] and Sharidan et al. [17], the similarity variables are introduced as

$$\eta = \left(\frac{c}{v_f} \right)^{1/2} y, \quad \tau = \omega t, \quad \psi = (cv_f)^{1/2} xf(\tau, \eta), \quad (6)$$

$$u = cxf'(\tau, \eta), \quad v = -(cv_f)^{1/2} f(\tau, \eta), \quad (7)$$

$$\theta(\tau, \eta) = \frac{T - T_\infty}{T_w - T_\infty}. \quad (8)$$

Applying this transformation to equations (1) to (5) we obtain the corresponding ordinary differential equations as

$$A_m f''' + C_1 (ff'' - f'^2) + C_2 \lambda G(\tau) \theta \cos \alpha - S_\sigma M f' = C_1 \Omega \frac{\partial f'}{\partial \tau}, \quad (9)$$

$$\left(\frac{k_{nf}}{k_f} + \frac{4}{3} R_d [1 + (\theta_w - 1) \theta]^3 \right) \theta'' + 4 R_d (\theta_w - 1) (1 + (\theta_w - 1) \theta)^2 (\theta')^2 \quad (10)$$

$$+ C_3 Pr (f \theta' - f' \theta) + Pr Ec \left[\frac{\mu_{nf}}{\mu_f} (f'')^2 + S_\sigma M (f')^2 \right] = C_3 Pr \Omega \frac{\partial \theta}{\partial \tau},$$

$$f(0) = 0, \quad f'(0) = 1, \quad \theta(0) = 1, \quad (11)$$

$$f'(\infty) \rightarrow 0, \quad \theta(\infty) \rightarrow 0. \quad (12)$$

where $S_\sigma = \sigma_{nf}/\sigma_f$, $R_d = \frac{4\sigma^* T_\infty^3}{k^* k_f}$, denotes the radiation parameter, $Pr = \frac{\mu_f (c_p)_f}{k_f}$, is the Prandtl number, $\lambda = \frac{Gr_x}{Re_x^2}$, is the mixed convection parameter with $Gr_x = g_0 \beta_{Tf} (T_w - T_\infty) x^3 / \nu_f^2$ and $Re_x = u_w x / \nu_f$ being the local Grashof and Reynolds numbers, respectively. For compactness, define $A_m = \frac{1}{(1-\phi)^{2.5}} \left(1 + \frac{1}{\beta} \right)$, $K_r = \frac{k_{nf}}{k_f} + \frac{4}{3} R_d$. Furthermore, the gravity modulation function is defined as $G(\tau) = 1 + \varepsilon \sin(\Omega \tau)$, where ε is the oscillation amplitude and Ω is the dimensionless frequency. This represents a harmonic oscillation about the mean gravitational acceleration g_0 , consistent with the g-jitter approach used by Sharidan et al. [17].

Table 2 lists the thermophysical properties of the CMC-water base fluid and copper nanoparticles, taken from the experimental and literature values reported in [9,18].

Table 2. Thermophysical Properties of Base Fluid and Nanoparticles, adopted from Freidoonimehr et al. [9] and Naseem et al. [18].				
Physical properties	C_p ($J \, kg^{-1} K^{-1}$)	ρ ($kg \, m^{-3}$)	k ($W \, m^{-1} K^{-1}$)	$\beta_T \times 10^5$ (K^{-1})
CMC-Water(< 0.4%)	4179	997.1	0.613	21
Nanoparticle (Cu)	385	8933	401	1.67

Of engineering interest are the wall shear and the radiation-corrected heat-transfer coefficient, given as:

$$C_f Re_x^{1/2} = 2 \left(1 + \frac{1}{\beta} \right) \frac{f''(0)}{(1 - \phi)^{2.5}}, \quad (13)$$

$$Nu^* = - \left(\frac{k_{nf}}{k_f} + \frac{4}{3} R_d \theta_w^3 \right) \theta'(0). \quad (14)$$

Where $\theta_w^3 = [1 + (\theta_w - 1)\theta(0)]^3$

3. Keller-box Numerical Method

The transformed system of equations (9)-(12) constitutes a coupled, nonlinear boundary value problem that cannot be solved analytically. For numerical solutions, we employ the Keller-box method, an implicit finite-difference technique widely recognized for its stability and second-order accuracy in solving parabolic boundary-layer equations. The method proceeds through four distinct stages: (i) reduction of the higher-order system to first-order form, (ii) discretization on a non-uniform grid using central differences at cell midpoints, (iii) linearization of the resulting algebraic equations via Newton's method, and (iv) efficient solution of the block-tridiagonal matrix system using the Thomas algorithm.

3.1 First-Order Reduction

The third-order momentum equation (9) and the second-order energy equation (10) are first reduced to an equivalent system of five first-order equations. We introduce the independent vector of dependent variables (f, u, v, θ, q) by defining:

$$u = f', u' = v, \text{ and } \theta' = q.$$

Substituting these relations into the transformed momentum and energy boundary-layer expressions yields:

$$A_m v' + C_1(fv - u^2) + C_2 \lambda G(\tau) \theta \cos \alpha - S_\sigma M u = C_1 \Omega \frac{\partial u}{\partial \tau}, \quad (15)$$

$$\frac{\partial}{\partial \eta} (K_{nr}(\theta) q) + C_3 Pr(fq - u\theta) + Pr Ec \left[\frac{\mu_{nf}}{\mu_f} (v)^2 + S_\sigma M (u)^2 \right] = C_3 Pr \Omega \frac{\partial \theta}{\partial \tau}, \quad (16)$$

Where the dynamic, temperature-dependent non-linear thermal diffusion parameter coefficient $K_{nr}(\theta)$ is explicitly extracted as:

$$K_{nr}(\theta) = \left(\frac{k_{nf}}{k_f} + \frac{4}{3} R_d [1 + (\theta_w - 1)\theta]^3 \right)$$

Expanding the spatial derivative in the energy framework results in the following computational form:

$$\begin{aligned} K_{nr}(\theta)q' + 4R_d(\theta_w - 1)(1 + (\theta_w - 1)\theta)^2 q^2 + C_3 Pr(fq - u\theta) \\ + PrEc \left[\frac{\mu_{nf}}{\mu_f} v^2 + S_\sigma M u^2 \right] = C_3 Pr \Omega \frac{\partial \theta}{\partial \tau}, \end{aligned} \quad (17)$$

The corresponding boundary conditions are written as:

$$f(\tau, 0) = 0, u(\tau, 0) = 1, \theta(\tau, 0) = 1, \quad \text{at } \eta = 0, \quad (18)$$

$$u(\tau, \eta_\infty) \rightarrow 0, \theta(\tau, \eta_\infty) \rightarrow 0, \quad \text{as } \eta \rightarrow \eta_\infty. \quad (19)$$

3.2 Space-Time Mesh Discretization

Let the rectangular computational domain in the (η, τ) plane be defined by a mesh network with grid coordinates given by:

$$\eta_0 = 0, \eta_j = \eta_{j-1} + h_j \quad (j = 1, 2, \dots, J), \eta_J = \eta_\infty,$$

$$\tau^0 = 0, \tau^n = \tau^{n-1} + \Delta\tau \quad (n = 1, 2, \dots, N),$$

where h_j is the spatial step size and $\Delta\tau$ is the time increment. Any general net function Z_j^n or spatial derivative evaluated at the localized grid midpoint segment $(\eta_{j-\frac{1}{2}}, \tau^{n-\frac{1}{2}})$ is approximated using averages and central differences:

$$Z_{j-\frac{1}{2}}^{n-\frac{1}{2}} = \frac{1}{2} (Z_j^n + Z_{j-1}^n + Z_j^{n-1} + Z_{j-1}^{n-1}),$$

$$\left(\frac{\partial Z}{\partial \tau}\right)_{j-\frac{1}{2}}^{n-\frac{1}{2}} = \frac{1}{2h_j} (Z_j^n - Z_{j-1}^n + Z_j^{n-1} - Z_{j-1}^{n-1})$$

$$\left(\frac{\partial Z}{\partial \tau}\right)_{j-\frac{1}{2}}^{n-\frac{1}{2}} = \frac{1}{2\Delta\tau} (Z_j^n + Z_{j-1}^n - Z_j^{n-1} - Z_{j-1}^{n-1})$$

Applying these centering approximations to the system of first-order formulations yields the discrete algebraic equivalent equations. To simplify, we group the known terms from the previous time step ($n - 1$) into a parameter constant denoted by $[R]_{j-\frac{1}{2}}^{n-1}$. The equations at the current time level n are written as:

$$\frac{f_j^n - f_{j-1}^n}{h_j} = u_{j-\frac{1}{2}}^n,$$

$$\frac{u_j^n - u_{j-1}^n}{h_j} = v_{j-\frac{1}{2}}^n,$$

$$\frac{\theta_j^n - \theta_{j-1}^n}{h_j} = q_{j-\frac{1}{2}}^n,$$

$$\begin{aligned} A_m \left(\frac{v_j^n - v_{j-1}^n}{h_j} \right) + C_1 \left[(fv)_{j-\frac{1}{2}}^n - (u^2)_{j-\frac{1}{2}}^n \right] + C_2 \lambda G(\tau^n) \theta_{j-\frac{1}{2}}^n \cos \alpha - S_\sigma M u_{j-\frac{1}{2}}^n \\ - \frac{C_1 \Omega}{\Delta\tau} \left(u_{j-\frac{1}{2}}^n \right) = [R1]_{j-\frac{1}{2}}^{n-1} \end{aligned} \quad (20)$$

$$\begin{aligned} K_{nr} \left(\theta_{j-\frac{1}{2}}^n \right) \left(\frac{q_j^n - q_{j-1}^n}{h_j} \right) + 4R_d(\theta_w - 1) \left[1 + (\theta_w - 1) \theta_{j-\frac{1}{2}}^n \right]^2 (q^2)_{j-\frac{1}{2}}^n \\ + C_3 Pr \left[(fq)_{j-\frac{1}{2}}^n - (u\theta)_{j-\frac{1}{2}}^n \right] + Pr Ec \left[\frac{\mu_{nf}}{\mu_f} (v^2)_{j-\frac{1}{2}}^n + S_\sigma M (u^2)_{j-\frac{1}{2}}^n \right] \\ - \frac{C_3 Pr \Omega}{\Delta\tau} \left(\theta_{j-\frac{1}{2}}^n \right) = [R2]_{j-\frac{1}{2}}^{n-1} \end{aligned} \quad (21)$$

3.3 Newton-Raphson Linearization

Because the algebraic system is highly non-linear, we introduce the iterative Newton-Raphson approximations at iteration level $(r + 1)$ by defining:

$$Z_j^{n,r+1} = Z_j^{n,r} + \delta Z_j^{n,r}$$

Substituting these expansions into the non-linear discrete equations and dropping the higher-order terms of small variations (δ^2) , we get a structured linear system of equations for the corrections:

$$\delta f_j - \delta f_{j-1} - \frac{h_j}{2} (\delta u_j + \delta u_{j-1}) = (r_1)_j,$$

$$\delta u_j - \delta u_{j-1} - \frac{h_j}{2} (\delta v_j + \delta v_{j-1}) = (r_2)_j,$$

$$\delta \theta_j - \delta \theta_{j-1} - \frac{h_j}{2} (\delta q_j + \delta q_{j-1}) = (r_3)_j$$

$$(a_1)_j \delta v_j + (a_2)_j \delta v_{j-1} + (a_3)_j \delta f_j + (a_4)_j \delta f_{j-1} + (a_5)_j \delta u_j + (a_6)_j \delta u_{j-1} + (a_7)_j \delta \theta_j + (a_8)_j \delta \theta_{j-1} = (r_4)_j, \quad (22)$$

$$(b_1)_j \delta q_j + (b_2)_j \delta q_{j-1} + (b_3)_j \delta f_j + (b_4)_j \delta f_{j-1} + (b_5)_j \delta u_j + (b_6)_j \delta u_{j-1} + (b_7)_j \delta \theta_j + (b_8)_j \delta \theta_{j-1} = (r_5)_j. \quad (23)$$

The coefficients $(b_1)_j$ through $(b_8)_j$ are updated to incorporate the non-linear thermal radiation terms:

$$(b_1)_j = \frac{K_{nr}}{h_j} + 4R_d(\theta_w - 1) \left[1 + (\theta_w - 1)\theta_{j-\frac{1}{2}} \right]^2 q_{j-\frac{1}{2}} + \frac{C_3 Pr}{2} f_{j-\frac{1}{2}}$$

$$(b_2)_j = -\frac{K_{nr}}{h_j} + 4R_d(\theta_w - 1) \left[1 + (\theta_w - 1)\theta_{j-\frac{1}{2}} \right]^2 q_{j-\frac{1}{2}} + \frac{C_3 Pr}{2} f_{j-\frac{1}{2}}$$

$$(b_7)_j = \frac{2R_d(\theta_w - 1)^2}{h_j} \left[1 + (\theta_w - 1)\theta_{j-\frac{1}{2}} \right]^2 (q_j - q_{j-1}) \\ + 4R_d(\theta_w - 1)^2 \left[1 + (\theta_w - 1)\theta_{j-\frac{1}{2}} \right] q_{j-\frac{1}{2}}^2 - \frac{C_3 Pr}{2} u_{j-\frac{1}{2}} - \frac{C_3 Pr \Omega}{2} 2\Delta\tau.$$

Here, $(r_1)_j$ to $(r_5)_j$ represent the current iteration residuals computed from the grid boundaries. Since all temperature and velocity terms in the non-linear functions are evaluated at the spatial midpoint $\eta_{j-1/2}$ via algebraic symmetric averages (e.g., $\theta_{j-1/2} = \frac{\theta_j + \theta_{j-1}}{2}$), the derivatives with respect to the node j are symmetric to node $j - 1$.

3.4 Block-Tridiagonal Solver

The linearized matrix correction equations are arranged in a block-tridiagonal form:

$$A_j \delta Y_{j-1} + B_j \delta Y_j = R_j.$$

where each matrix block component is a 5×5 matrix, and the vectors are defined as:

$$\delta Y_j = [\delta f_j \quad \delta u_j \quad \delta v_j \quad \delta \theta_j \quad \delta q_j]^T$$

This structured system is solved using a two-pass block elimination method (forward sweep and backward substitution). The numerical iteration continues until the maximum absolute error changes which satisfy the tolerance criteria:

$$\max\{|\delta v_0|, |\delta q_0|, \dots\} < 10^{-6}$$

This ensures that the non-linear heat transport parameters reach stable, invariant physical values.

4. Results and Discussion

All numerical computations presented in this section employ CMC-water as the base carrier fluid with copper nanoparticles as the dispersed solid phase. The default parameter values, unless explicitly varied, are maintained at $Pr = 6.2$, $\varepsilon = 0.5$, $\Omega = 0.2$, $M = 0.5$, $Rd = 0.5$, and $Ec = 0.01$. The Keller-box

algorithm was executed with a spatial domain extending to $\eta_{\infty} = 10$, with adaptive grid refinement near the wall to resolve the steep gradients characteristic of boundary-layer transport.

For presentation clarity, the plotted data were truncated at the visualization stage to exclude the far-field asymptotic tails, thereby ensuring that the physically significant boundary-layer behavior remains clearly visible without unnecessary scaling distortion.

4.1 Code Verification and Benchmarking

To establish confidence in the numerical implementation, the present model was systematically reduced to the classical Newtonian, non-radiative limiting configuration by setting $M = Rd = Ec = \varphi = \lambda = \varepsilon = \Omega = \alpha = 0$ and allowing $\beta \rightarrow \infty$. Under these restrictions, the governing equations reduce to the canonical mixed convection problem over a stretching sheet.

The left-hand columns of Table 3 present a quantitative comparison of the computed wall temperature gradient $-\theta'(0)$ against benchmark data from Sharidan et al. [17], Freidoonimehr et al. [9], and Rawi et al. [1]. This comparison is performed strictly under the reduced Newtonian, non-radiative limit defined above ($M = Rd = Ec = \varphi = \lambda = \varepsilon = \Omega = \alpha = 0, \beta \rightarrow \infty$). The close agreement across a broad spectrum of Prandtl numbers ($0.72 \leq Pr \leq 10$) confirms the accuracy and reliability of the present numerical framework. The maximum deviation from the reference solutions remains well within acceptable tolerances, validating both the discretization scheme and the convergence criteria employed.

As anticipated from thermal boundary-layer theory, Nu^* increases monotonically with Prandtl number, reflecting the reduced thermal diffusivity and consequently thinner thermal boundary layer at higher Pr values. The nanofluid case ($\varphi = 0.1$) consistently yields marginally higher Nu^* than the pure fluid case, with the enhancement becoming more pronounced at elevated Prandtl numbers. This behavior is attributed to the augmented effective thermal conductivity imparted by the highly conductive copper nanoparticles.

Table 3. Comparison of wall temperature gradient predictions with benchmark solutions from [1,9,17], computed under the reduced Newtonian, non-radiative limiting case ($M = Rd = Ec = \phi = \lambda = \varepsilon = \Omega = \alpha = 0, \beta \rightarrow \infty$).

Pr	Sharidan et al. [17]	Freidoonimehr et al. [9]	Rawi et al. [1]	Present $-\theta'(0)$	$Nu^*,$ $\phi = 0$	$Nu^*,$ $\phi = 0.1$
0.72	0.8086	0.8086	0.8086	0.80945	0.87527	0.87979
1.00	1.0000	1.0000	1.0000	1.00006	1.09947	1.10342
3.00	1.9238	1.9237	1.9239	1.92419	2.27015	2.34314
10.00	3.7225	3.7207	3.7230	3.72814	4.57325	4.83552

4.2 Influence of Casson Rheological Parameter

The impact of the Casson parameter β and the nanoparticle volume fraction ϕ on the streamwise velocity distribution $f'(\eta)$ is depicted in Figure 2. For fixed magnetic field strength and nanoparticle loading, increasing the Casson parameter (β) which corresponds to diminishing the yield stress influence—results in a systematic reduction of the velocity magnitude throughout the boundary layer. This behavior arises because smaller β values signify stronger non-Newtonian characteristics, specifically enhanced shear-thinning behavior that promotes momentum diffusion away from the wall. Conversely, as β increases toward the Newtonian limit, the velocity gradients sharpen, and the momentum boundary layer becomes thinner.

The addition of copper nanoparticles further modifies the velocity field through simultaneous alterations in the effective density and dynamic viscosity. The combined effect of particle loading and Casson rheology produces a monotonic decay of the velocity from the wall value $f'(0) = 1$ to the free-stream condition, consistent with the physical expectation that both magnetic damping and viscous resistance impede fluid motion.

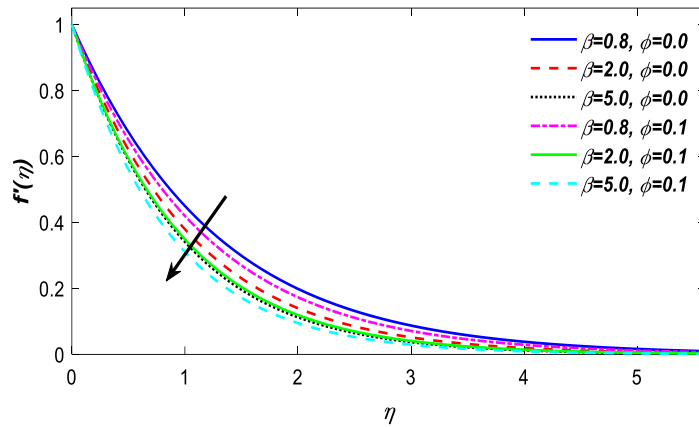


Fig. 2. Variation of the dimensionless velocity distribution $f'(\eta)$ with the Casson rheological parameter β and nanoparticle loading ϕ .

Figure 3 illustrates the corresponding temperature profiles $\theta(\eta)$ for varying β and ϕ . The temperature decays monotonically from the heated wall ($\theta = 1$) to the ambient isothermal condition ($\theta = 0$). For a prescribed magnetic interaction parameter, increasing the nanoparticle fraction elevates the temperature levels within the boundary layer. This thermal enhancement is directly attributable to the superior thermal conductivity of copper nanoparticles relative to the CMC-water base fluid, which facilitates more efficient heat conduction from the heated surface into the bulk flow. Interestingly, the Casson parameter exhibits a comparatively weaker influence on the temperature distribution than on the velocity field, suggesting that the thermal transport characteristics are governed primarily by the effective thermal properties rather than by the rheological modifications.

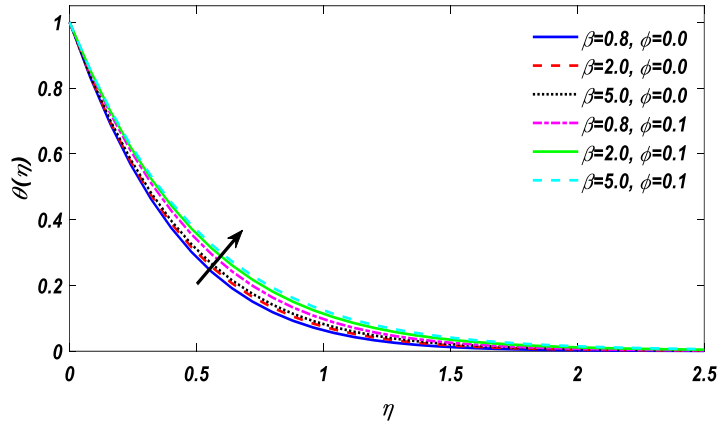


Fig.3. Thermal boundary layer structure $\theta(\eta)$ for different values of the Casson parameter and copper nanoparticle fraction.

4.3 Effect of Surface Inclination Angle

Figures 4 and 5 demonstrate the sensitivity of the velocity and temperature profiles to variations in the inclination angle α and the nanoparticle volume fraction ϕ . The buoyancy term in the momentum equation scales with $\cos \alpha$, thereby rendering the thermal driving force progressively weaker as the inclination angle increases from zero (vertical sheet) toward $\pi/2$ (horizontal sheet).

Consequently, at larger inclination angles, the buoyancy-assisted flow acceleration diminishes, and the velocity profile exhibits reduced magnitudes across the boundary layer. The temperature field, however, displays a slight elevation with increasing α , indicating that the reduced convective transport allows thermal energy to accumulate within the boundary layer, resulting in a modest thickening of the thermal region.

The nanoparticle loading effect persists across all inclination angles, with the copper particles consistently amplifying the temperature levels while concurrently suppressing the velocity magnitudes through enhanced viscous resistance.

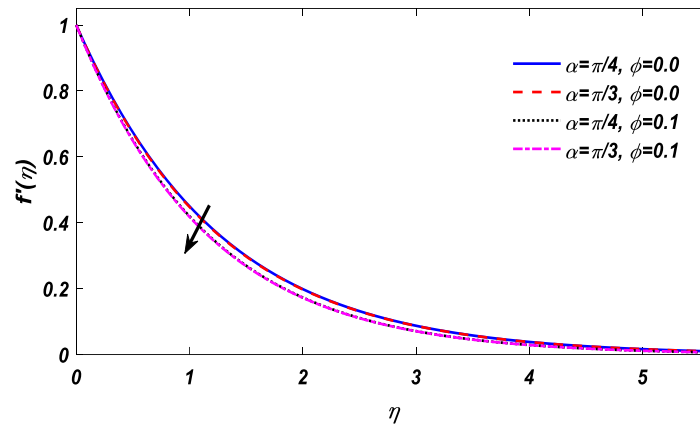


Fig. 4. Effect of inclination angle α and nanoparticle volume fraction ϕ on velocity profile $f'(\eta)$.

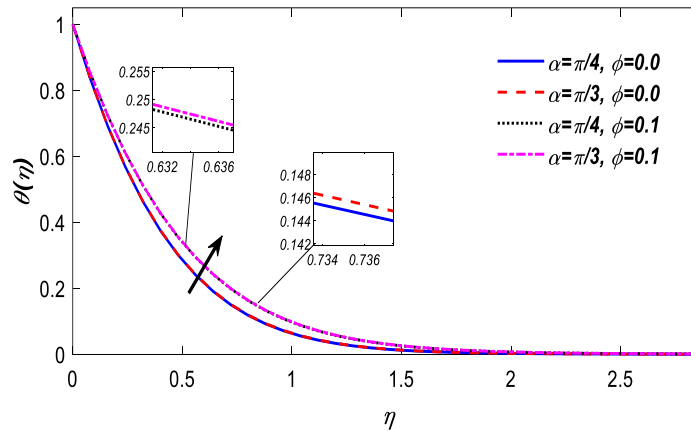


Fig. 5. Effect of inclination angle α and nanoparticle volume fraction ϕ on temperature profile $\theta(\eta)$.

4.4 Role of Nanoparticle Concentration

The parametric influence of the nanoparticle volume fraction ϕ is examined in Figures 6 and 7 for both the Casson fluid case ($\beta = 0.8$) and the Newtonian limiting case ($\beta \rightarrow \infty$). Increasing the particle loading produces a pronounced reduction in the velocity field, as the augmented effective density and viscosity of the suspension impose greater resistance to the stretching-induced flow. This observation is consistent

with the classical understanding that particle addition increases the momentum diffusivity, thereby thickening the hydrodynamic boundary layer.

In stark contrast, the temperature profile exhibits an upward shift with increasing ϕ , signifying enhanced thermal diffusion within the nanofluid. The highly conductive copper nanoparticles facilitate more efficient heat penetration into the fluid, effectively broadening the thermal boundary layer. This opposing behavior—velocity suppression versus temperature enhancement—underscores the competing mechanisms of momentum and thermal transport in nanoparticle suspensions. The comparison between the Casson and Newtonian cases reveals that the non-Newtonian rheology amplifies both effects: the velocity reduction is more severe, and the temperature rise is more pronounced in the Casson fluid than in its Newtonian counterpart, particularly at higher particle concentrations.

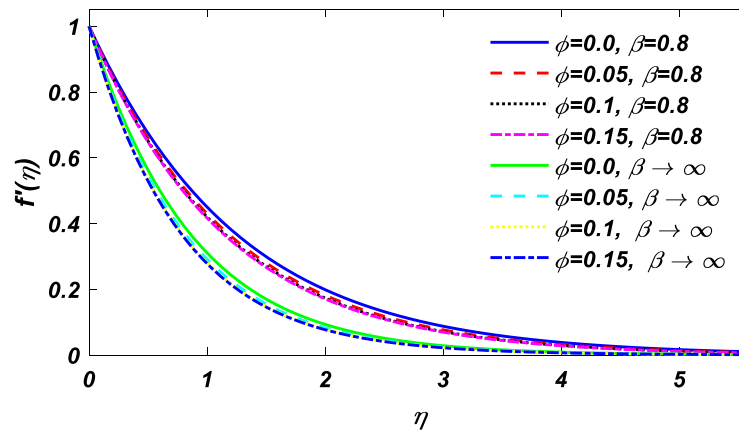


Fig. 6. Effect of nanoparticle volume fraction ϕ and Casson parameter β on velocity profile $f'(\eta)$.

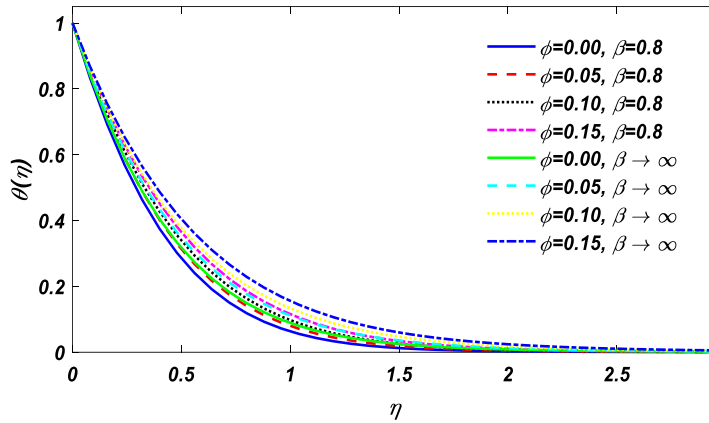


Fig. 7. Effect of nanoparticle volume fraction ϕ and Casson parameter β on temperature profile $\theta(\eta)$.

4.5 Engineering Wall Transport Quantities

Table 4 summarizes the computed wall-gradient parameters, specifically the skin friction coefficient (proportional to $f''(0)$) and the radiation-corrected Nusselt number Nu^* , for various β and α values. For the pure fluid case ($\phi = 0$), elevating β from 0.8 to 5 increases the wall shear magnitude from 0.751749 to 1.000113, indicating a stronger momentum transfer at the surface. This trend is mirrored in the nanofluid case ($\phi = 0.1$), where the corresponding $f''(0)$ values range from 0.832993 to 1.116372. The enhanced wall shear in the nanofluid arises from the combined effects of higher effective viscosity and reduced near-wall momentum diffusivity.

The radiation-corrected Nusselt number Nu^* , which incorporates both the conductive and radiative contributions to the heat flux, exhibits a systematic decrease with increasing β . This trend suggests that the modified velocity field resulting from Casson rheology alters the thermal boundary layer structure in a manner that slightly diminishes the overall heat transfer effectiveness.

Variations in the inclination angle from $\alpha = \pi/4$ to $\alpha = \pi/3$ produce a minor reduction in Nu , attributable to the weakened buoyancy component that partially drives the convective transport. The nanoparticle volume fraction, however, consistently enhances Nu despite producing a less negative raw temperature gradient $\theta'(0)$. This apparent paradox is resolved by recognizing that Nu^* includes the

multiplicative factor $Kr = \left(\frac{knf}{kf}\right) + (4/3)Rd$, which increases substantially with both ϕ and Rd . Consequently, the net radiation-corrected heat transfer coefficient remains elevated in the presence of nanoparticles, confirming the practical benefit of particle addition for thermal management applications.

Table 4. Computed wall shear stress and radiation-modified heat transfer coefficients for selected values of β and α .

Parameter	Value	$-f''(0),$ $\phi = 0$	$-\theta'(0),$ $\phi = 0$	$Nu^*,$ $\phi = 0$	$-f''(0),$ $\phi = 0.1$	$-\theta'(0),$ $\phi = 0.1$	$Nu^*,$ $\phi = 0.1$
β	0.8	-0.751749	-2.214694	3.691156	-0.832993	-1.963284	3.923245
β	2	-0.904553	-2.171463	3.619104	-1.006834	-1.914419	3.825598
β	5	-1.000113	-2.144476	3.574127	-1.116372	-1.884042	3.764894
α	$\pi/4$	-0.770577	-2.210567	3.684278	-0.848264	-1.959649	3.915981
α	$\pi/3$	-0.783968	-2.207558	3.679263	-0.859130	-1.956999	3.910687

4.6 Discussion of Physical Mechanisms

The collective observations from the parametric study elucidate several important physical mechanisms governing transport in the present magneto-radiative Casson nanofluid system. First, the magnetic field acts as a damping agent, suppressing the velocity field through the Lorentz force and indirectly influencing the thermal field by modifying the convective transport. Second, the Rosseland radiation mechanism, through its nonlinear dependence on the local temperature, introduces a temperature-dependent effective diffusivity that can significantly alter the thermal boundary-layer structure, particularly at elevated wall temperatures.

The interplay between Joule heating and viscous dissipation introduces additional thermal sources within the energy equation, contributing to the elevated temperature levels observed in the presence of magnetic and velocity gradients. These irreversible heating mechanisms, while often neglected in simpler models, can substantially affect the thermal performance of engineering systems operating under high-speed or high-field conditions.

Finally, the synergistic effects of nanoparticle loading, Casson rheology, and surface inclination produce a rich tapestry of transport phenomena that cannot be captured by models considering only a subset of

these mechanisms. The present unified framework provides a comprehensive tool for analyzing and optimizing thermal management systems that employ electrically conducting, non-Newtonian nanofluids under realistic operating conditions.

5. Conclusions

A comprehensive mathematical framework has been developed to describe the unsteady mixed convective transport of a magnetically responsive, radiatively active Casson-type Cu-CMC nanofluid over an inclined stretching surface with time-periodic gravitational modulation. The formulation encompasses transverse MHD resistance, Rosseland thermal radiation, viscous dissipation, Joule heating, nanoparticle volume fraction effects, surface inclination, and gravity oscillation. The governing boundary-layer equations, transformed through similarity variables, were solved numerically via the Keller-box finite-difference scheme. The principal findings of the investigation are summarized below.

Numerical Validation and Scheme Reliability: The Keller-box implementation successfully reproduces the benchmark solutions for the reduced Newtonian, non-radiative limiting configuration across a wide spectrum of Prandtl numbers. The close agreement with previously published data [1, 9, 17] confirms the accuracy, stability, and robustness of the computational approach.

Casson Rheological Effects: Elevating the Casson parameter—corresponding to diminished yield stress influences the wall shear stress magnitude while producing a modest reduction in the radiation-corrected Nusselt number. This behavior is observed for both the pure CMC fluid and the Cu-CMC nanofluid, although the magnitudes of the changes are amplified in the particle-laden case due to the modified effective viscosity.

Nanoparticle Loading Influence: Dispersing copper nanoparticles in the CMC-water base fluid produces dual opposing effects: the velocity profile is suppressed throughout the boundary layer due to augmented density and viscosity, while the temperature distribution is elevated because of enhanced effective thermal conductivity. This contrasting behavior underscores the competing transport mechanisms operating in nanoparticle suspensions.

Inclination Angle Sensitivity: Increasing the surface inclination angle relative to the vertical reduces the buoyancy component that drives the flow, resulting in lower velocity magnitudes and a slight elevation of

the temperature field. The radiation-corrected heat transfer coefficient exhibits a corresponding, albeit modest, decrease with increasing inclination.

Combined Transport Mechanisms: The simultaneous action of magnetic damping, thermal radiation, viscous dissipation, and Joule heating generates physically consistent velocity and thermal profiles across the entire boundary-layer domain. The nonlinear Rosseland radiation treatment, incorporating the full temperature dependence of the radiative flux, provides a more accurate representation of the thermal field than the linearized approximation commonly employed in the literature.

Heat Transfer Enhancement: The radiation-corrected Nusselt number Nu^* increases with nanoparticle loading despite the raw temperature gradient $\theta'(0)$ becoming less negative. This seemingly counterintuitive result is resolved by recognizing that Nu^* incorporates the combined conductive-radiative factor $Kr = \left(\frac{kn_f}{k_f}\right) + \left(\frac{4}{3}\right)Rd$, which grows substantially with both the radiation parameter and the nanoparticle concentration.

Practical Implications: The quantitative results presented in this study provide guidance for the design and optimization of thermal management systems that employ electrically conducting non-Newtonian nanofluids. The parametric trends identified herein can inform the selection of appropriate operating conditions, nanoparticle loadings, and magnetic field strengths for specific engineering applications.

Future Research Directions: The present model can be extended in several promising directions to incorporate additional physical complexity. Potential avenues for future investigation include the effects of nonlinear radiation with variable absorption coefficients, temperature-dependent viscosity and thermal conductivity, velocity slip and thermal jump at the solid boundary, entropy generation analysis for second-law optimization, homogeneous and heterogeneous chemical reactions, and the use of hybrid nanoparticle combinations for enhanced thermal performance.

Nomenclature			
Symbol	Meaning	Symbol	Meaning
u, v	velocity components along x and y directions	θ	dimensionless temperature
T	fluid temperature	η	similarity variable
T_w, T_∞	wall and ambient temperatures	τ	dimensionless time
f	dimensionless stream function	ϕ	nanoparticle volume fraction
β	Casson parameter	Pr	Prandtl number
M	magnetic parameter	Ω	dimensionless frequency
R_d	radiation parameter	λ	mixed convection parameter
Ec	Eckert number	Nu^*	radiation-corrected heat-transfer coefficient

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References

- [1]. Rawi, N.A., Ilias, M.R., Lim, Y.J., Isa, Z.M. and Shafie, S., 2017, September. Unsteady mixed convection flow of Casson fluid past an inclined stretching sheet in the presence of nanoparticles. In *Journal of Physics: Conference Series* (Vol. 890, No. 1, p. 012048). IOP Publishing.
- [2]. Sharada, K. and Shankar, B., 2015. MHD mixed convection flow of a Casson fluid over an exponentially stretching surface with the effects of sores, dufour, thermal radiation and chemical reaction. *World Journal of Mechanics*, 5(9), pp.165-177.
- [3]. Ullah, I., Bhattacharyya, K., Shafie, S. and Khan, I., 2016. Unsteady MHD mixed convection slip flow of Casson fluid over nonlinearly stretching sheet embedded in a porous medium with chemical reaction, thermal radiation, heat generation/absorption and convective boundary conditions. *PloS one*, 11(10), p.e0165348.
- [4]. Ullah, I., Khan, I. and Shafie, S., 2016. Hydromagnetic Falkner-Skan flow of Casson fluid past a moving wedge with heat transfer. *Alexandria Engineering Journal*, 55(3), pp.2139-2148.

- [5]. Sohail, M., Chu, Y.M., El-Zahar, E.R., Nazir, U. and Naseem, T., 2021. Contribution of joule heating and viscous dissipation on three dimensional flow of Casson model comprising temperature dependent conductance utilizing shooting method. *Physica Scripta*, 96(8), p.085208.
- [6]. Prasad, K.V., Vajravelu, K., Vaidya, H., Basha, N.Z. and Umesh, V., 2018. Thermal and species concentration of MHD Casson fluid at a vertical sheet in the presence variable fluid properties. *Ain Shams Engineering Journal*, 9(4), pp.1763-1779.
- [7]. Choi, S.U., 1995, November. Enhancing thermal conductivity of fluids with nanoparticles. In *ASME International Mechanical Engineering Congress and Exposition* (Vol. 17421, pp. 99-105). American Society of Mechanical Engineers.
- [8]. Parasuraman, L., Peddisetty, N.C. and Periyannagounder, G., 2015. Radiation effects on an unsteady MHD natural convective flow of a nanofluid past a vertical plate. *Thermal Science*, 19(3), p.1037.
- [9]. Freidoonimehr, N., Rashidi, M.M. and Mahmud, S., 2015. Unsteady MHD free convective flow past a permeable stretching vertical surface in a nano-fluid. *International Journal of Thermal Sciences*, 87, pp.136-145.
- [10]. Hayat, T., Bilal, A.M., Shehzad, S.A. and Alsaedi, A., 2015. Mixed convection flow of Casson nanofluid over a stretching sheet with convectively heated chemical reaction and heat source/sink. *Journal of Applied Fluid Mechanics*, 8(4), pp.801-810.
- [11]. Ullah, I., Khan, I. and Shafie, S., 2016. MHD natural convection flow of Casson nanofluid over nonlinearly stretching sheet through porous medium with chemical reaction and thermal radiation. *Nanoscale Research Letters*, 11(1), p.527.
- [12]. Naseem, T. and Shahzad, A., 2023. Thermal transport in nanofluid across a radiated permeable sheet with irreversible effects based on the shape of the particles. *International Journal of Numerical Methods for Heat & Fluid Flow*, 33(6), pp.2073-2099.
- [13]. Lin, Y., Zheng, L., Zhang, X., Ma, L. and Chen, G., 2015. MHD pseudo-plastic nanofluid unsteady flow and heat transfer in a finite thin film over stretching surface with internal heat generation. *International Journal of Heat and Mass Transfer*, 84, pp.903-911.
- [14]. Westra, J.G., 1989. Rheology of carboxymethyl cellulose with xanthan gum properties. *Macromolecules*, 22(1), pp.367-370.
- [15]. Abdelrahim, K.A. and Ramaswamy, H.S., 1995. High temperature/pressure rheology of carboxymethyl cellulose (CMC). *Food Research International*, 28(3), pp.285-290.

- [16]. Yaşar, F., Toğrul, H. and Arslan, N., 2007. Flow properties of cellulose and carboxymethyl cellulose from orange peel. *Journal of Food Engineering*, 81(1), pp.187-199.
- [17]. Sharidan, S., Amin, N. and Pop, I., 2006. g-Jitter mixed convection adjacent to a vertical stretching sheet. *Microgravity-Science and Technology*, 18(1), pp.5-14.
- [18]. Naseem, T., Mebarek-Oudina, F., Shahzad, A., Fish, H. and Hamida, M.B., 2026. Enhanced thermodynamic irreversibility and heat transfer in rotating graphene oxide-water nanofluids over a stretching surface. *Modern Physics Letters B*, 40(03), p.2550267.
- [19]. Hakeem, A.A., S., P., Bhose, G. and Sivanandam, S., 2024. Magneto-convective hybrid nanofluid slip flow over a moving inclined thin needle in a Darcy-Forchheimer porous medium with viscous dissipation. *International Journal of Numerical Methods for Heat & Fluid Flow*, 34(1), pp.334-352.
- [20]. Nadeem, S., Haq, R.U. and Khan, Z.H., 2014. Numerical solution of non-Newtonian nanofluid flow over a stretching sheet. *Applied Nanoscience*, 4(5), pp.625-631.
- [21]. Kuznetsov, A.V. and Nield, D.A., 2010. Natural convective boundary-layer flow of a nanofluid past a vertical plate. *International Journal of Thermal Sciences*, 49(2), pp.243-247.
- [22]. Khan, W.A. and Pop, I., 2010. Boundary-layer flow of a nanofluid past a stretching sheet. *International Journal of Heat and Mass Transfer*, 53(11-12), pp.2477-2483.
- [23]. Mahapatra, T.R. and Gupta, A.S., 2002. Heat transfer in stagnation-point flow towards a stretching sheet. *Heat and Mass Transfer*, 38(6), pp.517-521.
- [24]. Zainal, N.A., Nazar, R., Naganthran, K. and Pop, I., 2021. MHD flow and heat transfer of hybrid nanofluid over a permeable moving surface in the presence of thermal radiation. *International Journal of Numerical Methods for Heat & Fluid Flow*, 31(3), pp.858-879.
- [25]. Turkyilmazoglu, M., 2011. Multiple solutions of heat and mass transfer of MHD slip flow for the viscoelastic fluid over a stretching sheet. *International Journal of Thermal Sciences*, 50(11), pp.2264-2276.
- [26]. Brewster, M.Q., 1992. *Thermal Radiative Transfer and Properties*. John Wiley & Sons.
- [27]. Rashidi, M.M., Freidoonimehr, N., Hosseini, A., Bég, O.A. and Hung, T.K., 2014. Homotopy simulation of nanofluid dynamics from a non-linearly stretching isothermal permeable sheet with transpiration. *Meccanica*, 49(2), pp.469-482.
- [28]. Mukhopadhyay, S., 2013. Casson fluid flow and heat transfer over a nonlinearly stretching surface. *Chinese Physics B*, 22(7), p.074701.

- [29]. Qasim, M., Hayat, T. and Obaidat, S., 2012. Radiation effect on the mixed convection flow of a viscoelastic fluid along an inclined stretching sheet. *Zeitschrift für Naturforschung A-Journal of Physical Sciences*, 67(3), p.195.
- [30]. Naseem, T. and Shahzad, A., 2025. Water-based nanofluid flow and heat transfer in thin film over an unsteady stretching sheet: An entropy analysis. *Numerical Heat Transfer, Part A: Applications*, 86(5), pp.1201-1225.
- [31]. Shahzad, A., Gulistan, U., Ali, R., Iqbal, A., Benim, A.C., Kamran, M., Khan, S.U.D., Khan, S.U.D. and Farooq, A., 2020. Numerical study of axisymmetric flow and heat transfer in a liquid film over an unsteady radially stretching surface. *Mathematical Problems in Engineering*, 2020(1), p.6737243.
- [32]. Shahzad, A., Liaqat, F., Ellahi, Z., Sohail, M., Ayub, M. and Ali, M.R., 2022. Thin film flow and heat transfer of Cu-nanofluids with slip and convective boundary condition over a stretching sheet. *Scientific Reports*, 12(1), p.14254.
- [33]. Abel, M.S., Datti, P.S. and Mahesha, N., 2009. Flow and heat transfer in a power-law fluid over a stretching sheet with variable thermal conductivity and non-uniform heat source. *International Journal of Heat and Mass Transfer*, 52(11-12), pp.2902-2913.
- [34]. Alharbi, K.A.M., Ashraf, W., Eldin, S.M., Yassen, M.F. and Jamshed, W., 2023. Applied heat transfer modeling in conventional hybrid ($\text{Al}_2\text{O}_3\text{-CuO}$)/ $\text{C}_2\text{H}_6\text{O}_2$ and modified-hybrid nanofluids ($\text{Al}_2\text{O}_3\text{-CuO-Fe}_3\text{O}_4$)/ $\text{C}_2\text{H}_6\text{O}_2$ between slippery channel by using least square method (LSM). *AIMS Mathematics*, 8(2), pp.4321-4341.