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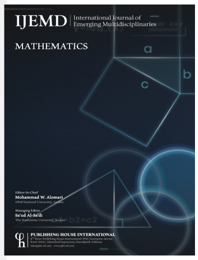

The Simpson's inequality for r–convex Mappings

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Abstract

For an absolutely continuous mapping $f''' : I \subseteq \mathbb{R} \to \mathbb{R}$, on I°, where $a,b \in I$ with $a < b$. It is proved that, if $\left|f^{(4)}\right|$ is convex on $[a,b]$, then inequality

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{241920}\left(23\left|f^{(4)}(a)\right|+19\left|f^{(4)}(b)\right|\right).$$

holds.

Keywords: *Simpson's inequality, bounded variation, r-convex.*
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1. Introduction

Suppose $f : [a,b] \to \mathbb{R}$ is fourth times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_{\infty} := \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. The following inequality

$$\left|\int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \le \frac{(b-a)^5}{2880}\left\|f^{(4)}\right\|_{\infty} \tag{1.1}$$

holds, and it is well known in the literature as Simpson's inequality. It is well known that if the mapping f is neither four times differentiable nor is the fourth derivative $f^{(4)}$ bounded on (a,b), then we cannot apply the classical Simpson quadrature formula.

In [11], Pečarić and Varošanec, obtained some inequalities of Simpson's type for functions whose n-th derivative, $n \in \{0,1,2,3\}$ is of bounded variation, as follow:

Theorem 1.1. *Let $n \in \{0,1,2,3\}$. Let f be a real function on $[a,b]$ such that $f^{(n)}$ is function of bounded variation. Then*

$$\left|\int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \le C_n (b-a)^{n+1} \bigvee_a^b \left(f^{(n)}\right), \tag{1.2}$$

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where,

$$C_0 = \frac{1}{3},\ C_1 = \frac{1}{24},\ C_2 = \frac{1}{324},\ C_3 = \frac{1}{1152},$$

and $\bigvee_a^b \left(f^{(n)}\right)$ *is the total variation of* $f^{(n)}$ *on the interval* $[a,b]$.

It is convenient to note that, the inequality (1.1) with $n = 0$, was proved in literature by Dragomir [6]. Also, Ghizzetti and Ossicini [10], proved that if f''' is an absolutely continuous mapping with total variation $\bigvee_a^b (f)$, then (1.2) with $n = 3$ holds.
In recent years many authors were established several generalizations of the Simpson's inequality for mappings of bounded variation and for Lipschitzian, monotonic, and absolutely continuous mappings via kernels, for refinements, counterparts, generalizations and several Simpson's type inequalities see [2]–[18].

A positive function f is log-convex on a real interval $[a,b]$ if for all $x,y \in [a,b]$ and $\lambda \in [0,1]$ we have

$$f(\lambda x + (1-\lambda)y) \le f^{\lambda}(x) f^{1-\lambda}(y). \tag{1.3}$$

If the reverse inequality holds, f is said to be log-concave.
In addition, the power mean $M_r(x,y;\lambda)$ of order r of positive numbers x,y is defined by

$$M_r(x,y;\lambda) = \begin{cases} (\lambda x^r + (1-\lambda) y^r)^{1/r}, & r \neq 0 \\ x^{\lambda} y^{1-\lambda}, & r = 0 \end{cases} \tag{1.4}$$

In the special case $\lambda = \frac{1}{2}$ we contract this notation to $M_r(x,y)$. In view of the above, a natural generalizing concept is that of r-convexity (see [5]).

Definition 1.2. *A positive function* $f : [a,b] \to \mathbb{R}_+$*, is called r-convex function if for all* $x,y \in [a,b]$ *and* $\lambda \in [0,1]$ *we have*

$$f(\lambda x + (1-\lambda)y) \le M_r(f(x), f(y); \lambda) \tag{1.5}$$

In the above definition, we have that 0-convex functions are simply log-convex functions and 1-convex functions are ordinary convex functions. Also, Definition 1.2 of r-convexity can be expanded as the condition that

$$f^r(\lambda x + (1-\lambda)y) \le \begin{cases} \lambda f^r(x) + (1-\lambda) f^r(y), & r \neq 0 \\ f^{\lambda}(x) f^{1-\lambda}(y), & r = 0 \end{cases}$$

In 1998, Pearce et. al. [12], proved that for a nonnegative function f that possesses a second derivative. If $r \ge 2$, then

$$\frac{d^2 f^r}{dx^2} = r(r-1) f^{r-2} (f')^2 + r f^{r-1} f''$$

which is nonnegative if $f'' \ge 0$. Hence under the above restrictions, ordinary convexity implies r-convexity. The reverse implication is not the case, as is shown by the function, $f(x) = x^{1/2}$ for $x > 0$.

Recently, Alomari [1] proved the following inequality which is of Simpson's type for quasi-convex mappings.

Theorem 1.3. *Let* $f''' : I \subseteq \mathbb{R} \to \mathbb{R}$ *be an absolutely continuous mapping on* I° *such that* $f^{(4)} \in L[a,b]$*, where* $a,b \in I$ *with* $a < b$*. If* $\left|f^{(4)}\right|$ *is quasi-convex on* $[a,b]$*, then the following inequality holds:*

$$\left| \int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \right| \le \frac{(b-a)^5}{5760}\left[\sup\left\{\left|f^{(4)}(a)\right|, \left|f^{(4)}\left(\frac{a+b}{2}\right)\right|\right\} + \sup\left\{\left|f^{(4)}\left(\frac{a+b}{2}\right)\right|, \left|f^{(4)}(b)\right|\right\}\right]. \tag{1.6}$$

In this paper, we obtain an inequality of Simpson type via r-convex mappings. This approach allows us to investigate Simpson's quadrature rule that have restrictions on the behavior of the integrand and thus to deal with larger classes of functions. In general, we show that our result is better than the previous result obtained in the literature.

2. Inequalities of Simpson's type for r-convex functions

In order to prove our main results, we start with the following lemma (see [7]):

Lemma 2.1. *Let $f''':I\subseteq\mathbb{R}\to\mathbb{R}_+$ be an absolutely continuous mapping on I°, where $a,b\in I$ with $a<b$. If $f^{(4)}\in L[a,b]$, then the following equality holds:*

$$\int_a^b f(x)\,dx-\frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]=(b-a)^5\int_0^1 p(t)\,f^{(4)}(ta+(1-t)b)\,dt, \tag{2.1}$$

where,

$$p(t)=\begin{cases}\frac{1}{24}t^3\left(t-\frac{2}{3}\right), & t\in\left[0,\frac{1}{2}\right]\\ \frac{1}{24}(t-1)^3\left(t-\frac{1}{3}\right), & t\in\left(\frac{1}{2},1\right]\end{cases}.$$

Therefore, we may state our main result as follows:

Theorem 2.2. *Let $f''':I\subseteq\mathbb{R}\to\mathbb{R}_+$ be an absolutely continuous mapping on I° such that $f^{(4)}\in L[a,b]$, where $a,b\in I$ with $a<b$. If $\left|f^{(4)}\right|$ is r-convex, $r>1$ on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\le\frac{(b-a)}{24}\left[\frac{\left(2^{-\frac{r+1}{r}}+3r\cdot2^{-\frac{1}{r}}+4\right)\cdot r^4}{(1+9r+26r^2+24r^3)(1+11r+30r^2)}\left|f^{(4)}(a)\right|\right.\\&\qquad\left.+\frac{\left(2^{-\frac{r+1}{r}}+15r\cdot2^{-\frac{r+1}{r}}+27r^2\cdot2^{-\frac{1}{r}}-24r^2+16r\right)\cdot r^4}{720r^6+1764r^5+1624r^4+735r^3+175r^2+21r+1}\left|f^{(4)}(b)\right|\right]\end{aligned} \tag{2.2}$$

Proof. From Lemma 2.1, and since f is s-convex, we have

$$\begin{aligned}&\left|\int_a^b f(x)\,dx-\frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\&=\left|(b-a)^5\int_0^1 p(t)\,f^{(4)}(ta+(1-t)b)\,dt\right|\\&\le(b-a)^5\int_0^1|p(t)|\left|f^{(4)}(ta+(1-t)b)\right|dt\\&=(b-a)^5\int_0^{1/2}|p(t)|\left|f^{(4)}(ta+(1-t)b)\right|dt\\&\qquad+(b-a)^5\int_{1/2}^1|p(t)|\left|f^{(4)}(ta+(1-t)b)\right|dt\\&\le\frac{(b-a)}{24}\int_0^{1/2}\left|t^3\left(t-\tfrac{2}{3}\right)\right|\left(t\left|f^{(4)}(a)\right|^r+(1-t)\left|f^{(4)}(b)\right|^r\right)^{1/r}dt\\&\qquad+\frac{(b-a)}{24}\int_{1/2}^1\left|(t-1)^3\left(t-\frac{1}{3}\right)\right|\left(t\left|f^{(4)}(a)\right|^r+(1-t)\left|f^{(4)}(b)\right|^r\right)^{1/r}dt.\end{aligned}$$

Using the fact that $\sum_{i=1}^n(a_i+b_i)^k\le\sum_{i=1}^n a_i^k+\sum_{i=1}^n b_i^k$, for $0<k<1$, $a_1,a_2,...,a_n\ge0$ and $b_1,b_2,...,b_n\ge0$, we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\le \frac{(b-a)}{24}\int_0^{1/2}\left|t^3\left(t-\tfrac{2}{3}\right)\right|\left(t\left|f^{(4)}(b)\right|^r+(1-t)\left|f^{(4)}(a)\right|^r\right)^{1/r}dt \\
&\quad+\frac{(b-a)}{24}\int_{1/2}^1\left|(t-1)^3\left(t-\frac{1}{3}\right)\right|\left(t\left|f^{(4)}(b)\right|^r+(1-t)\left|f^{(4)}(a)\right|^r\right)^{1/r}dt \\
&\le \frac{(b-a)}{24}\int_0^{1/2}\left|t^3\left(t-\tfrac{2}{3}\right)\right|\left(t^{\frac{1}{r}+1}\left|f^{(4)}(a)\right|+t(1-t)^{\frac{1}{r}}\left|f^{(4)}(b)\right|\right)dt \\
&\quad+\frac{(b-a)}{24}\int_{1/2}^1\left|(t-1)^3\left(t-\frac{1}{3}\right)\right|\left(t^{\frac{1}{r}+1}\left|f^{(4)}(a)\right|+t(1-t)^{\frac{1}{r}}\left|f'(b)\right|\right)dt \\
&= \frac{(b-a)}{24}\left[\frac{\left(2^{-\frac{r+1}{r}}+3r\cdot 2^{-\frac{1}{r}}+4\right)\cdot r^4}{(1+9r+26r^2+24r^3)(1+11r+30r^2)}\left|f^{(4)}(a)\right|\right. \\
&\quad\left.+\frac{\left(2^{-\frac{r+1}{r}}+15r\cdot 2^{-\frac{r+1}{r}}+27r^2\cdot 2^{-\frac{1}{r}}-24r^2+16r\right)\cdot r^4}{720r^6+1764r^5+1624r^4+735r^3+175r^2+21r+1}\left|f^{(4)}(b)\right|\right],
\end{aligned}$$

which gives the required result and the proof is completed. □

Corollary 2.3. *In Theorem 2.2, choose $r=1$, the result holds for convex functions, and we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{241920}\left(23\left|f^{(4)}(a)\right|+19\left|f^{(4)}(b)\right|\right). \tag{2.3}$$

Moreover, If $\left\|f^{(4)}\right\|_\infty := \sup_{x\in(a,b)}\left|f^{(4)}(x)\right| < \infty$, then we have the inequality

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{5760}\left\|f^{(4)}\right\|_\infty. \tag{2.4}$$

Remark 2.4. *We note that, the constant in (2.4) improves the constant in (1.6).*

3. Conclusion

In this article we improve the costant in the celebrated Simpson's inequality as shown in Corollary 2.3.

Competing Interests

The author(s) declare no competing interests.

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