IJEMD-M, 1 (2) (2022), 23 - 34 https://doi.org/10.54938/ijemdm.2022.01.2.65

International Journal of Emerging Multidiciplinaries: Mathematics

Research Paper
Journal Homepage: www.ijemd.com
ISSN: 2790-1998 (print), 2790-3257 (online)

Various Series Concerning the Zeta Function

Vuk Stojiljković

University Of Novi Sad , Novi Sad , Serbia

Abstract

In this paper we evaluated various series concerning the ζ function. We also have shown how our Lemma can be paired up with different generating functions to produce more series as a consequence.

Keywords: *Series, Riemann Zeta function, integration*
2010 Mathematics Subject Classification: *33E20, 33B99*

1. Introduction

Sums have been an ongoing topic of investigation since their introduction. The earliest record of people summing is the formula for the sum of the first n numbers. From then on people developed more complicated sums. Even the great Euler concerned himself with evaluating the famous sum, the basel problem. Many sums have been discovered since then, see the following books about sums [2], [3], [8], [10]. Many papers have been written about them, see [14], [6], [13]. The type of sums we will investigate today can be found in the book [12]. The first known definition is as follows.

Definition 1.1. *The polylogarithm, see [11] is defined by a power series in z, given by*

$$\mathrm{Li}_s(z) = \sum_{k=1}^{+\infty} \frac{z^k}{k^s}.$$

This definition is valid for arbitrary complex order s and for all complex arguments z with $|z| < 1$. We will also need the definition given by

$$\mathrm{Li}_s(z) = \int_0^z \frac{\mathrm{Li}_{s-1}(z)}{z} dz.$$

For $z = 1$ we get the Riemann zeta function ζ which is also a function of complex variable s. For more information see [4], [5], [9].

$$\mathrm{Li}_s(1) = \zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \Re(s) > 1$$

The second definition is given.

Definition 1.2. *The gamma function is defined by a convergent improper integral, for $\Re(z) > 0$, see [1]*

$$\Gamma(z) = \int_0^{+\infty} x^{z-1} e^{-x} dx.$$

Email addresses: **vuk.stojiljkovic999@gmail.com**

The following relations hold.

$$\Gamma(z+1) = \Gamma(z)$$
$$\Gamma(z+1) = z!$$

We give our third definition.

Definition 1.3. *We define the following sequence*

$$\zeta_s(k) = \sum_{l=1}^{k} \frac{1}{l^s}.$$

We give our first crucial Lemma in this paper.

Lemma 1.4. *The following equalities hold*

$$\zeta(2m) - \zeta_{2m}(k) = -\frac{1}{(2m-1)!} \int_0^1 \frac{x^k \ln^{2m-1}(x)}{1-x} dx$$
$$\zeta(2m+1) - \zeta_{2m+1}(k) = \frac{1}{(2m)!} \int_0^1 \frac{x^k \ln^{2m}(x)}{1-x} dx$$

Proof. Both of the equalities can be derived from the fact that the integral

$$\int_0^1 x^k \ln^m(x) dx = \frac{(-1)^m \Gamma(1+m)}{(1+k)^{1+m}}$$

gives different results depending on whether m is even or odd. If m is odd we have a minus and an odd factorial in the result, therefore we divide the integral with it to get the positive value $\frac{1}{(k+1)^{2m}}$. If m is even we get a positive sign with an even factorial, therefore diving the integral with it gives us the clear $\frac{1}{(k+1)^{2m+1}}$ form. Adding $\frac{1}{1-x}$ to both even and odd forms and expanding it into a series makes the series continue to form a tail of the ζ series. □

We give our second Lemma.

Lemma 1.5. *The following equality holds for $|y|, |z| < 1$, see [14]*

$$\sum_{k=1}^{+\infty} y^k \left(\mathrm{Li}_s(z) - z - .... - \frac{z^k}{k^s} \right) = \frac{1}{y-1} (\mathrm{Li}_s(zy) - zy) - \frac{y}{y-1} (\mathrm{Li}_s(z) - z)$$

2. Main results

We give our first Theorem using the ζ tail representation.

Theorem 1.6. *The following equality holds for $|z| < 1$*

$$\sum_{k=1}^{+\infty} \left(\mathrm{Li}_1(z) - z - ... - \frac{z^k}{k} \right) (\zeta(2) - \zeta_2(k))$$
$$= \frac{1}{6(z-1)} \left(6(z-1)\mathrm{Li}_2\left(\frac{1}{1-z}\right) + 6\mathrm{Li}_2\left(\frac{z-1}{z}\right) - 2\pi^2 z + 6z\log^2(1-z) \right)$$
$$+ \frac{1}{6(z-1)} \left(-6\log^2(1-z) + 3\log^2\left(\frac{1}{z}\right) - \pi^2\log(1-z) + \pi^2 z\log(1-z) \right)$$
$$+ \frac{1}{6(z-1)} \left(6\log\left(\frac{1}{z}\right)\log(1-z) - 6z\log(-z)\log(1-z) \right)$$
$$+ \frac{1}{6(z-1)} \left(6\log(-z)\log(1-z) + 2\pi^2 \right)$$

Proof. Using Lemma 1.4 , the part with even zeta function and setting $m = 1$ we get that the representation is as follows

$$\zeta(2) - \zeta_2(k) = -\frac{1}{(2-1)!}\int_0^1 \frac{x^k \ln(x)}{1-x}dx$$

Using it in our case gives us the following

$$\sum_{k=1}^{+\infty}\left(\mathrm{Li}_1(z) - z - \ldots - \frac{z^k}{k}\right)\left(\zeta(2) - \frac{1}{1^2} - \ldots - \frac{1}{k^2}\right)$$

$$= \sum_{k=1}^{+\infty}\left(\mathrm{Li}_1(z) - z - \ldots - \frac{z^k}{k}\right)\left(-\frac{1}{(2-1)!}\int_0^1 \frac{x^k \ln(x)}{1-x}dx\right)$$

$$= -\int_0^1 \sum_{k=1}^{+\infty} x^k \left(\mathrm{Li}_1(z) - z - \ldots - \frac{z^k}{k}\right)\frac{\ln(x)}{1-x}dx$$

For the inside summation we use Lemma 1.5 which gives us the following

$$= -\int_0^1 \left(\frac{1}{x-1}(\mathrm{Li}_1(xz) - xz) - \frac{x}{x-1}(\mathrm{Li}_1(z) - z)\right)\cdot\frac{\ln(x)}{1-x}dx$$

$$= \frac{1}{6(z-1)}\left(6(z-1)\,\mathrm{Li}_2\left(\frac{1}{1-z}\right) + 6\,\mathrm{Li}_2\left(\frac{z-1}{z}\right) - 2\pi^2 z + 6z\log^2(1-z)\right)$$

$$+\frac{1}{6(z-1)}\left(-6\log^2(1-z) + 3\log^2\left(\frac{1}{z}\right) - \pi^2\log(1-z) + \pi^2 z\log(1-z)\right)$$

$$+\frac{1}{6(z-1)}\left(6\log\left(\frac{1}{z}\right)\log(1-z) - 6z\log(-z)\log(1-z)\right)$$

$$+\frac{1}{6(z-1)}\left(6\log(-z)\log(1-z) + 2\pi^2\right)$$

The Theorem is proved. □

Corollary 1.7. *Using the representation from the last Theorem and setting $z = \frac{1}{2}$ we get the following equality*

$$\sum_{k=1}^{+\infty}\left(\mathrm{Li}_1\left(\frac{1}{2}\right) - \ldots - \frac{\frac{1}{2^k}}{k}\right)(\zeta(2) - \zeta_2(k)) = \log^2(2) - \frac{1}{12}\pi^2(\log(4) - 1)$$

The next Theorem utilizes Lemma 1.4 and allows us to evaluate various series containing tails of the ζ function.

Theorem 1.8. *The following equalities holds for $|y| < 1$*

$$\sum_{k=0}^{+\infty} y^k(\zeta(2m) - \zeta_{2m}(k)) = \frac{\mathrm{Li}_{2m}(y) - \zeta(2m)}{(y-1)}$$

$$\sum_{k=0}^{+\infty} y^k(\zeta(2m+1) - \zeta_{2m+1}(k)) = \frac{(\mathrm{Li}_{2m+1}(y) - \zeta(2m+1))}{(y-1)}$$

Proof. Both of the equalities can be proved in a similar way. Use Lemma 1.4 to rewrite the partial sums of the Riemann Zeta function , exchange the sum and integral sign and apply the geometric series formula. We are left with

$$\sum_{k=0}^{+\infty} y^k(\zeta(2m) - \zeta_{2m}(k)) = -\frac{1}{(2m-1)!}\int_0^1 \frac{1}{1-xy}\frac{\ln^{2m-1}(x)}{1-x}dx$$

$$\sum_{k=0}^{+\infty} y^k(\zeta(2m+1) - \zeta_{2m+1}(k)) = \frac{1}{(2m)!}\int_0^1 \frac{1}{1-xy}\frac{\ln^{2m}(x)}{1-x}dx$$

We will solve the first integral, the second one is analogous. Let us consider the first integral

$$-\frac{1}{(2m-1)!}\int_0^1 \frac{1}{1-xy}\frac{\ln^{2m-1}(x)}{1-x}dx.$$

Let us call the constant in the front c, to minimize the clutter in the formulas. Introduce a substitution $x=e^{-k}$, from which we get

$$-\frac{1}{(2m-1)!}\int_0^1 \frac{1}{1-xy}\frac{\ln^{2m-1}(x)}{1-x}dx = c\int_0^{+\infty}\frac{1}{1-e^{-k}y}\frac{(-k)^{2m-1}}{1-e^{-k}}(e^{-k})dk$$

$$-\frac{1}{(2m-1)!}\int_0^1 \frac{1}{1-xy}\frac{\ln^{2m-1}(x)}{1-x}dx = c\cdot(-1)^{2m-1}\int_0^{+\infty}\frac{e^k}{e^k-y}\frac{k^{2m-1}}{e^k-1}e^k(e^{-k})dk$$

Cancelling what we can and calling the constant $(-1)^{2m-1}$ in front of the integral b, to minimize the clutter in the formulas, we proceed as follows

$$cb\int_0^{+\infty}\frac{e^k}{e^k-y}\frac{k^{2m-1}}{e^k-1}e^k(e^{-k})dk = cb\int_0^{+\infty}\frac{e^k}{e^k-y}\frac{k^{2m-1}}{e^k-1}dk$$

Performing partial fractions in the following way $\frac{e^k}{(e^k-y)(e^k-1)} = \frac{y}{(y-1)(e^k-y)} + \frac{1}{(e^k-1)(1-y)}$ and applying it to the integral, we get

$$cb\int_0^{+\infty}\frac{e^k}{e^k-y}\frac{k^{2m-1}}{e^k-1}dk = cb\int_0^{+\infty}k^{2m-1}\left(\frac{y}{(y-1)(e^k-y)}+\frac{1}{(e^k-1)(1-y)}\right)dk$$

We will focus on solving the first integral, second one is the special case of the first one.

$$cb\cdot\frac{y}{y-1}\int_0^{+\infty}\frac{k^{2m-1}}{(e^k-y)}dy = cb\cdot\frac{y}{y-1}\int_0^{+\infty}\frac{e^{-k}k^{2m-1}}{1-ye^{-k}}dk = cb\cdot\frac{y}{y-1}\int_0^{+\infty}\sum_{n=0}^{+\infty}y^n k^{2m-1}e^{-k(n+1)}dk$$

Switching the places of the sum and the integral and integrating the integrand, we get

$$\frac{cb}{y-1}\cdot\Gamma(2m)\sum_{n=0}^{+\infty}\frac{y^{n+1}}{(n+1)^{2m}} = \frac{cb}{y-1}\Gamma(2m)\operatorname{Li}_{2m}(y)$$

Second integral is the same and the result can be obtained by setting $y=1$ in the summation of the first integral. Therefore we get

$$-\frac{1}{(2m-1)!}\int_0^1 \frac{1}{1-xy}\frac{\ln^{2m-1}(x)}{1-x}dx = \frac{(-1)^{2m}}{(2m-1)!}\left(\frac{\operatorname{Li}_{2m}(y)\Gamma(2m)}{y-1}+\frac{\zeta(2m)\Gamma(2m)}{1-y}\right)$$

Which gives us the equality.

$$-\frac{1}{(2m-1)!}\int_0^1 \frac{1}{1-xy}\frac{\ln^{2m-1}(x)}{1-x}dx = \frac{\operatorname{Li}_{2m}(y)-\zeta(2m)}{(y-1)}$$

The Theorem is proved. □

Corollaries of the previous Theorem are now given.

Corollary 1.9. *The following equalities hold.*
Setting $m=2$ and $y=\frac{1}{5}$ in the first portion of Theorem 1.8 we get

$$\textbf{\textit{a)}}\sum_{k=0}^{+\infty}\left(\frac{1}{5}\right)^k(\zeta(4)-\zeta_4(k)) = \frac{\pi^4}{72}-\frac{5\operatorname{Li}_4\left(\frac{1}{5}\right)}{4}$$

b) *Setting $m=1$ and integrating the second equality from 0 to y we get*

$$\sum_{k=0}^{+\infty}\frac{y^{k+1}}{k+1}(\zeta(3)-\zeta_3(k)) = \log(1-y)(\operatorname{Li}_3(y)-\zeta(3))+\frac{\operatorname{Li}_2(y)^2}{2}$$

Proof.
$$\sum_{k=0}^{+\infty} \frac{y^{k+1}}{k+1}\left(\zeta(3)-\zeta_3(k)\right)=\int_0^y \frac{(\mathrm{Li}_3(y)-\zeta(3))}{(y-1)}dy$$
We will separate the integrals and solve them as indefinite ones, then put in the boundaries.
$$=\int \frac{\mathrm{Li}_3(y)}{(y-1)}dy-\int \frac{\zeta(3)}{(y-1)}dy$$
Focusing on to the first integral
$$\int \frac{\mathrm{Li}_3(y)}{y-1}dy=-\int \frac{\mathrm{Li}_3(y)}{1-y}dy$$
Integrating by parts, taking
$$u=-\mathrm{Li}_3(y), du=-\frac{\mathrm{Li}_2(y)}{y}, dv=\frac{1}{1-y}dy, v=-\ln(1-y)$$
$$-\int \frac{\mathrm{Li}_3(y)}{1-y}dy=\mathrm{Li}_3(y)\ln(1-y)-\int \frac{\ln(1-y)\,\mathrm{Li}_2(y)}{y}dy$$
Now let us focus to solve the integral we got
$$\int \frac{\ln(1-y)\,\mathrm{Li}_2(y)}{y}dy$$
Integrating by parts, taking
$$u=\mathrm{Li}_2(y), du=-\frac{\ln(1-y)}{y}, dv=\frac{\ln(1-y)}{y}dy, v=-\mathrm{Li}_2(\mathrm{y})$$
$$\int \frac{\ln(1-y)\,\mathrm{Li}_2(y)}{y}dy=-\mathrm{Li}_2^2(y)-\int \frac{\ln(1-y)\,\mathrm{Li}_2(y)}{y}dy$$
From which we get
$$\int \frac{\ln(1-y)\,\mathrm{Li}_2(y)}{y}dy=-\frac{\mathrm{Li}_2^2(y)}{2}$$
Putting this back we get
$$-\int \frac{\mathrm{Li}_3(y)}{1-y}dy=\mathrm{Li}_3(y)\ln(1-y)-\left(-\frac{\mathrm{Li}_2^2(y)}{2}\right)=\mathrm{Li}_3(y)\ln(1-y)+\frac{\mathrm{Li}_2^2(y)}{2}$$
The second integral is trivial
$$-\int \frac{\zeta(3)}{y-1}dy=-\ln(1-y)\zeta(3)$$
Combining the integrals we get
$$\int_0^y \frac{(\mathrm{Li}_3(y)-\zeta(3))}{(y-1)}dy=\mathrm{Li}_3(y)\ln(1-y)+\frac{\mathrm{Li}_2^2(y)}{2}-\ln(1-y)\zeta(3)$$
Putting the boundaries from 0 to y we get
$$\left(\mathrm{Li}_3(y)\ln(1-y)+\frac{\mathrm{Li}_2^2(y)}{2}-\ln(1-y)\zeta(3)\right)\Bigg|_0^y$$
When $y\to 0$ we get 0 and when $y\to y$ we get the same expression, therefore we obtain the result
$$\sum_{k=0}^{+\infty} \frac{y^{k+1}}{k+1}\left(\zeta(3)-\zeta_3(k)\right)=\log(1-y)(\mathrm{Li}_3(y)-\zeta(3))+\frac{\mathrm{Li}_2(y)^2}{2}$$
$\square$

Setting $y=\frac{1}{2}$ we get the equality

$$\sum_{k=0}^{+\infty}\left(\frac{1}{2}\right)^{k+1}\frac{\zeta(3)-\zeta_3(k)}{k+1}$$
$$=\frac{1}{288}\left(\pi^4+12\pi^2\ln^2 2-12\ln^4 2+36\ln 2\zeta(3)\right)$$

***c**) Setting $m=1$ and integrating the first equality in Theorem 1.8 from 0 to y we get*

$$\sum_{k=0}^{+\infty}\frac{y^{k+1}}{k+1}\left(\zeta(2)-\zeta_2(k)\right)$$
$$=-2\operatorname{Li}_3(1-y)-\operatorname{Li}_2(y)\log(1-y)+\frac{1}{6}\log(1-y)\left(\pi^2-6\log(1-y)\log(y)\right)+2\zeta(3)$$

The proof is similar to the last one, therefore is omitted and left as an exercise to an interested reader. Setting $y=\frac{1}{2}$ we get the equality

$$\sum_{k=0}^{+\infty}\left(\frac{1}{2}\right)^{k+1}\frac{\zeta(2)-\zeta_2(k)}{k+1}=\frac{1}{12}\left(\pi^2\ln 2+2\ln^3 2+3\zeta(3)\right)$$

The following Theorem shows how other generating functions can be paired up with the tails of the ζ function.

Theorem 1.10. *The following equality holds,*

$$\sum_{k=1}^{+\infty}\frac{\zeta(2)-\zeta_2(k)}{k+1}=2\zeta(3)-\zeta(2)$$

Proof. Let us consider the following expansion

$$\sum_{k=1}^{+\infty}\frac{x^k}{k+1}=-\frac{x+\ln(x)}{x}$$

Adding the tail of $\zeta(2)$ and using Lemma 1.4 we get

$$\sum_{k=1}^{+\infty}\frac{x^k}{k+1}\left(\zeta(2)-\zeta_2(k)\right)=\sum_{k=1}^{+\infty}\frac{x^k}{k+1}\left(-\int_0^1\frac{y^k\ln(y)}{1-y}dy\right)$$

Interchanging places between the integral and the sum is allowed since the sum is a power series, from which we get that

$$\sum_{k=1}^{+\infty}\frac{x^k}{k+1}\left(\zeta(2)-\zeta_2(k)\right)=\int_0^1-\sum_{k=1}^{+\infty}\frac{(xy)^k}{k+1}\frac{\ln(y)}{1-y}dy$$
$$=\int_0^1\frac{xy+\ln(1-xy)}{xy}\frac{\ln(y)}{1-y}dy$$

Setting $x=1$ and proceeding, we get

$$\sum_{k=1}^{+\infty}\frac{\zeta(2)-\zeta_2(k)}{k+1}=\int_0^1\frac{\ln(y)\left(y+\ln(1-y)\right)}{y(1-y)}dy$$

Splitting the integrand and using partial fractions we get

$$=\int_0^1\frac{\ln(y)\ln(1-y)}{y}dy+\int_0^1\frac{\ln(y)\ln(1-y)}{1-y}dy+\int_0^1\frac{y\ln(y)}{y(1-y)}dy$$

Introducing a substitution in the second integral $1-y=t$ we get that it equals the first integral, and the third one is easily seen to be $-\frac{\pi^2}{6}$.

$$= 2\int_0^1 \frac{\ln(y)\ln(1-y)}{y}dy - \frac{\pi^2}{6}$$

The first integral can be solved expanding the $\ln(1-y)$ into a series and solving the remaining integral to get the ζ function. Therefore we get the equality

$$\sum_{k=1}^{+\infty} \frac{\zeta(2)-\zeta_2(k)}{k+1} = 2\zeta(3)-\zeta(2).$$

The Theorem is proved. □

The following Theorem is really beautiful because it connects the ζ function and rational expression with the simple constant 1.

Theorem 1.11. *The following equaltiy holds*

$$\sum_{k=0}^{+\infty} \frac{\zeta(2)-\zeta_2(k)}{(k+1)(k+2)} = 1$$

Proof. Let us observe the following equality which holds for $[-1,0)\cup(0,1]$

$$\sum_{k=0}^{+\infty} \frac{x^k}{(k+1)(k+2)} = \frac{x - x\ln(1-x)+\ln(1-x)}{x^2}$$

Adding the tail of the zeta function to it, we get the following

$$\sum_{k=0}^{+\infty} \frac{x^k(\zeta(2)-\zeta_2(k))}{(k+1)(k+2)}$$

Writing the tail of the zeta function as the integral using Lemma 1.4 we get

$$\sum_{k=0}^{+\infty} \frac{x^k}{(k+1)(k+2)} \cdot \left(-\int_0^1 \frac{y^k\ln(y)}{1-y}dy\right)$$

Exchanging the places of the sum and the integral, which is allowed since the sum is a power series, we get

$$= -\int_0^1 \sum_{k=0}^{+\infty} \frac{(yx)^k}{(k+1)(k+2)} \frac{\ln(y)}{1-y}dy = -\int_0^1 \frac{xy - xy\ln(1-xy)+\ln(1-xy)}{x^2y^2}\frac{\ln(y)}{1-y}dy$$

Setting $x=1$ we get

$$\sum_{k=0}^{+\infty} \frac{\zeta(2)-\zeta_2(k)}{(k+1)(k+2)} = -\int_0^1 \frac{y-y\ln(1-y)+\ln(1-y)}{y^2}\frac{\ln(y)}{1-y}dy$$

We will solve the integral as an indefinite integral and then evaluate it at $y=1$ and $y=0$.
Separating each integral we get

$$-\int_0^1 \frac{y-y\ln(1-y)+\ln(1-y)}{y^2}\frac{\ln(y)}{1-y}dy = -\int \frac{\ln(y)}{y(1-y)}dy + \int \frac{\ln(1-y)\ln(y)}{y(1-y)}dy - \int \frac{\ln(y)\ln(1-y)}{y^2(1-y)}dy$$

Focusing on the first integral and using partial fractions we get

$$-\int \frac{\ln(y)}{y(1-y)} = -\int \frac{\ln(y)}{y}dy - \int \frac{\ln(y)}{1-y}dy$$

The first integral is easily done by a substitution and second one is a polylogarithm

$$-\int \frac{\ln(y)}{y(1-y)}dy = -\frac{1}{2}\ln^2(y) - \mathrm{Li}_2(1-y)$$

Focusing on the second integral and using partial fractions we get

$$\int \frac{\ln(1-y)\ln(y)}{y(1-y)}dy = \int \frac{\ln(1-y)\ln(y)}{y}dy + \int \frac{\ln(y)\ln(1-y)}{1-y}dy$$

First one is easily done by integration by parts from which we get

$$\int \frac{\ln(1-y)\ln(y)}{y}dy = -\ln(y)\,\mathrm{Li}_2(y) + \mathrm{Li}_3(y)$$

Second one is similarly done with partial integration from which we get

$$\int \frac{\ln(y)\ln(1-y)}{1-y}dy = \ln(1-y)\,\mathrm{Li}_2(1-y) - \mathrm{Li}_3(1-y)$$

Now focusing on to the third integral and using partial fractions we get

$$-\int \frac{\ln(y)\ln(1-y)}{y^2(1-y)}dy = -\int \frac{\ln(1-y)\ln(y)}{y}dy - \int \frac{\ln(1-y)\ln(y)}{y^2}dy - \int \frac{\ln(1-y)\ln(y)}{1-y}dy$$

First and third integral are the ones we already obtained for the second integral, therefore we focus our attention to the second integral. Performing partial integration taking $u = -\frac{\ln(1-y)}{y}$ and $dv = \frac{\ln(y)}{y}$ we get

$$-\int \frac{\ln(1-y)\ln(y)}{y^2}dy = -\frac{\ln(1-y)\ln^2(y)}{2y} - \frac{1}{2}\int \frac{\ln(1-y)\ln^2(y)}{y^2}dy - \frac{1}{2}\int \frac{\ln^2(y)}{y(1-y)}dy$$

Focusing on to the first integral and doing partial integration taking $u = \ln(1-y)$ and $dv = \frac{\ln^2(y)}{y^2}$ we get

$$\int \frac{\ln(1-y)\ln^2(y)}{y^2}dy = -\ln(1-y)\left(\frac{\ln^2(y)+2\ln(y)+2}{y}\right) - \int \frac{1}{1-y}\cdot\frac{\ln^2(y)+2\ln(y)+2}{y}dy$$

Plugging this back into the integral we were solving we get

$$-\int \frac{\ln(1-y)\ln(y)}{y^2}dy$$
$$= -\frac{\ln(1-y)\ln^2(y)}{2y} - \frac{1}{2}\left(-\ln(1-y)\left(\frac{\ln^2(y)+2\ln(y)+2}{y}\right) - \int \frac{\ln^2(y)}{y(1-y)} - 2\int \frac{\ln(y)}{y(1-y)}dy - 2\int \frac{1}{y(1-y)}dy\right)$$
$$-\frac{1}{2}\int \frac{\ln^2(y)}{y(1-y)}dy$$

Now we observe that integrals $\frac{1}{2}\int \frac{\ln^2(y)}{y(1-y)}dy - \frac{1}{2}\int \frac{\ln^2(y)}{y(1-y)}dy$ cancel eachother out, and we are left with

$$-\int \frac{\ln(1-y)\ln(y)}{y^2}dy = -\frac{\ln(1-y)\ln^2(y)}{2y} + \frac{1}{2}\ln(1-y)\left(\frac{\ln^2(y)+2\ln(y)+2}{y}\right) + \int \frac{\ln(y)}{y(1-y)}dy + \int \frac{dy}{y(1-y)}$$

The first integral has already been done and the last one is trivial, which at the end gives us the result

$$-\int \frac{\ln(1-y)\ln(y)}{y^2}dy$$
$$= -\frac{\ln(1-y)\ln^2(y)}{2y} + \frac{1}{2}\ln(1-y)\left(\frac{\ln^2(y)+2\ln(y)+2}{y}\right) + \frac{1}{2}\ln^2(y) + \mathrm{Li}_2(1-y) + \ln(y) - \ln(1-y)$$

When we combine all the the solved integrals, simplify the expression and put in the boundaries we are left with

$$-\int_0^1 \frac{y-y\ln(1-y)+\ln(1-y)}{y^2}\frac{\ln(y)}{1-y}dy =$$

$$= \left(-\frac{\log(1-y)\log^2(y)}{2y} + \frac{\log(1-y)\left(\log^2(y)+2\log(y)+2\right)}{2y} + \log(y) - \log(1-y) \right)\Bigg|_0^1$$

Since

$$\lim_{y\to 1} -\frac{\log(1-y)\log^2(y)}{2y} + \frac{\log(1-y)\left(\log^2(y)+2\log(y)+2\right)}{2y} + \log(y) - \log(1-y) = 0$$

And

$$\lim_{y\to 0} -\frac{\log(1-y)\log^2(y)}{2y} + \frac{\log(1-y)\left(\log^2(y)+2\log(y)+2\right)}{2y} + \log(y) - \log(1-y) = -1$$

We get that

$$\sum_{k=0}^{+\infty} \frac{\zeta(2)-\zeta_2(k)}{(k+1)(k+2)} = -\int_0^1 \frac{y-y\ln(1-y)+\ln(1-y)}{y^2}\frac{\ln(y)}{1-y}dy = 0-(-1) = 1$$

And we finally obtain the equality

$$\sum_{k=0}^{+\infty} \frac{\zeta(2)-\zeta_2(k)}{(k+1)(k+2)} = 1.$$

The Theorem is proved. □

Corollary 1.12. *Using a similar reasoning like in the previous Theorem, the following equality can be obtained, setting $x=-1$ instead of $x=1$ before integrating*

$$\sum_{k=0}^{+\infty} \frac{(-1)^k(\zeta(2)-\zeta_2(k))}{(k+1)(k+2)} = \frac{1}{4}\left(-2(\zeta(3)+2)+\pi^2(\log(4)-1)+\log(256)\right)$$

The next Theorem connects Fibonacci numbers with the tail of the ζ function.

Theorem 1.13. *The following equality holds for $|x| < \frac{1}{\phi}$, where $\phi = \frac{1+\sqrt{5}}{2}$*

$$\sum_{k=0}^{+\infty} x^k F_k(\zeta(2)-\zeta_2(k))$$

$$= -\frac{3\left(\left(\sqrt{5}-5\right)x-2\sqrt{5}\right)\mathrm{Li}_2\left(\frac{2x}{\sqrt{5}-1}\right) - 3\left(\left(\sqrt{5}+5\right)x-2\sqrt{5}\right)\mathrm{Li}_2\left(-\frac{2x}{\sqrt{5}+1}\right) + 5\pi^2 x}{30\left(x^2+x-1\right)}.$$

Proof. We begin by recalling the generating function for the Fibonacci numbers, see [7]

$$\sum_{k=0}^{+\infty} x^k F_k = \frac{x}{1-x-x^2}$$

Adding the tail of the ζ series to the sum we get the following

$$\sum_{k=0}^{+\infty} x^k F_k(\zeta(2)-\zeta_2(k))$$

Now using Lemma 1.4 while setting $m=1$ to rewrite the tail of the ζ function as the integral

$$\sum_{k=0}^{+\infty} x^k F_k(\zeta(2)-\zeta_2(k)) = \sum_{k=0}^{+\infty} x^k F_k \cdot \left(-\int_0^1 \frac{y^k\ln(y)}{1-y}dy\right)$$

Switching places of the sum and the integral and using the generating function of the Fibonacci numbers we get the following equality

$$\sum_{k=0}^{+\infty} x^k F_k\left(\zeta(2)-\zeta_2(k)\right)=\int_0^1-\frac{(xy)\log(y)}{(1-y)\left(-x^2y^2-xy+1\right)}\,dy$$

Writing $1-xy-x^2y^2$ in the form $(y-y_1)(y-y_2)$ and then doing partial fractions

$$\frac{y}{(1-y)(1-xy-x^2y^2)}$$

$$=\frac{4x^2}{(-2x+\sqrt{5}-1)(2x+\sqrt{5}+1)(y-1)}-\frac{2(\sqrt{5}-1)x^2}{\sqrt{5}(-2x+\sqrt{5}-1)(2xy-\sqrt{5}+1)}+\frac{2(1+\sqrt{5})x^2}{\sqrt{5}(2x+\sqrt{5}+1)(2xy+\sqrt{5}+1)}$$

and integrating term by term, we get

$$\sum_{k=0}^{+\infty} x^k F_k\left(\zeta(2)-\zeta_2(k)\right)$$

$$=-\frac{3\left(\left(\sqrt{5}-5\right)x-2\sqrt{5}\right)\mathrm{Li}_2\left(\frac{2x}{\sqrt{5}-1}\right)-3\left(\left(\sqrt{5}+5\right)x-2\sqrt{5}\right)\mathrm{Li}_2\left(-\frac{2x}{\sqrt{5}+1}\right)+5\pi^2x}{30\left(x^2+x-1\right)}.$$

The proof is complete. □

In the following we give a corollary of the previously derived Theorem.

Corollary 1.14. *The following equalities hold*
a)*Setting $x=\frac{1}{2}$ in the previously derived Theorem, we obtain the equality*

$$\sum_{k=0}^{+\infty}\frac{F_k}{2^k}\left(\zeta(2)-\zeta_2(k)\right)$$

$$=\frac{2}{15}\left(3\left(\frac{1}{2}\left(\sqrt{5}-5\right)-2\sqrt{5}\right)\mathrm{Li}_2\left(\frac{1}{\sqrt{5}-1}\right)\right)$$

$$+\frac{2}{15}\left(-3\left(\frac{1}{2}\left(\sqrt{5}+5\right)-2\sqrt{5}\right)\mathrm{Li}_2\left(-\frac{1}{\sqrt{5}+1}\right)+\frac{5\pi^2}{2}\right).$$

b) *Setting $x=\frac{1+\sqrt{5}}{2^{p+1}}$ and $x=\frac{1-\sqrt{5}}{2^{p+1}}, p\geqslant 2$ respectfully in the derived Theorem, we obtain*

$$\sum_{k=0}^{+\infty}\left(\frac{1+\sqrt{5}}{2^{p+1}}\right)^k F_k\left(\zeta(2)-\zeta_2(k)\right)$$

$$=\frac{2^{p-1}\left(12\sqrt{5}\left(2^p+1\right)\mathrm{Li}_2\left(2^{-p-1}\left(\sqrt{5}+3\right)\right)-6\left(\sqrt{5}2^{p+1}-3\sqrt{5}-5\right)\mathrm{Li}_2\left(-2^{-p}\right)-5\pi^2\left(\sqrt{5}+1\right)\right)}{15\left(2^p+1\right)\left(-2^{p+1}+\sqrt{5}+3\right)}$$

$$\sum_{k=0}^{+\infty}\left(\frac{1-\sqrt{5}}{2^{p+1}}\right)^k F_k\left(\zeta(2)-\zeta_2(k)\right)$$

$$=\frac{2^{p-1}\left(-6\left(\sqrt{5}2^{p+1}-3\sqrt{5}+5\right)\mathrm{Li}_2\left(-2^{-p}\right)+12\sqrt{5}\left(2^p+1\right)\mathrm{Li}_2\left(-2^{-p-1}\left(\sqrt{5}-3\right)\right)-5\pi^2\left(\sqrt{5}-1\right)\right)}{15\left(2^p+1\right)\left(2^{p+1}+\sqrt{5}-3\right)}$$

Subtracting the second from the first series and multiplying by $\frac{1}{\sqrt{5}}$ *we get the following*

$$\sum_{k=0}^{+\infty} F_k \frac{(\zeta(2)-\zeta_2(k))\left(\left(\frac{1+\sqrt{5}}{2^{p+1}}\right)^k-\left(\frac{1-\sqrt{5}}{2^{p+1}}\right)^k\right)}{\sqrt{5}}$$

Taking $\frac{1}{2^p}$ *in the front, we are left with*

$$\sum_{k=0}^{+\infty} F_k \frac{(\zeta(2)-\zeta_2(k))\left(\left(\frac{1+\sqrt{5}}{2}\right)^k-\left(\frac{1-\sqrt{5}}{2}\right)^k\right)}{\sqrt{5}} \cdot \frac{1}{2^{kp}}$$

Observing that the expression inside the brackets is a Fibonacci sequence, $F_k = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k-\left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}}$, *we obtain the following equality*

$$\sum_{k=0}^{+\infty} F_k^2 \frac{(\zeta(2)-\zeta_2(k))}{2^{kp}}$$

$$= \frac{1}{15\sqrt{5}\,(2^p+1)\,(-3\;2^p+4^p+1)} 2^{p-1}\left(5\sqrt{5}(2^p-1)\pi^2-12\sqrt{5}(1-3\cdot 2^p+4^p)\operatorname{Li}_2\left(2^{-p}\right)\right)$$

$$+\frac{1}{15\sqrt{5}\,(2^p+1)\,(-3\;2^p+4^p+1)} 2^{p-1}\left(6\sqrt{5}(1-3\cdot 2^p+4^p)\operatorname{Li}_2\left(4^{-p}\right)\right)$$

$$-\frac{1}{15\sqrt{5}\,(2^p+1)\,(-3\;2^p+4^p+1)} 2^{p-1}\left(3(1+2^p)(-5-3\sqrt{5}+2^{p+1}\cdot\sqrt{5})\operatorname{Li}_2\left(-2^{-1-p}(-3+\sqrt{5})\right)\right)$$

$$-\frac{1}{15\sqrt{5}\,(2^p+1)\,(-3\;2^p+4^p+1)} 2^{p-1}\left(3(1+2^p)(5-3\sqrt{5}+2^{1+p}\cdot\sqrt{5})\operatorname{Li}_2\left(2^{-1-p}(3+\sqrt{5})\right)\right)$$

Setting $p=2$ *in the expression we got, we obtain*

$$\sum_{k=0}^{+\infty} F_k^2 \frac{(\zeta(2)-\zeta_2(k))}{2^{2k}} \sim 0.221583$$

As a consequence, we can obtain many series of the Fibonacci squared-zeta type using the formula above.

3.Conclusion

1. We verified all the numerical results via `Wolfram Alpha`.
2. In this paper we evaluated various ζ series of the form found in the new book [12]. We also proved a useful Lemma for the integral representation of the tail of the ζ function.
3. Questions arise whether tails of other special functions can be found in terms of the integral representations.

Competing Interests

The author declares no competing interests.

References

[1] Abramowitz M., Stegun I.A., *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*, Dover Books on Mathematics, 1964.
[2] Bonar D.D and Kourym.M.J., *Real infinite series*, MAA, Washington DC, 2006.
[3] Davis, H.T. 2015., *The summation of series (Dover Books on Mathematics)*, Mineola, New York: Dover Publications. ISBN-13: 63 978-0486789682.
[4] Edwards M.H., *Riemann's zeta function*, Dover Publications; Dover Ed edition, June 13, 2001.

[5] Fabiano, N. Zeta function and some of its properties, *Vojnotehnički glasnik/Military Technical Courier*, **68**(4), 895-906 (2020).
[6] Furdui, O. Two surprising series with harmonic numbers and the tail of zeta(2), *Gazeta Matematica, Seria A*, **33**(112)(1-2),1-8 (2005).
[7] Grimaldi. R, *Fibonacci and catalan numbers: An introduction*, Wiley, March 13, 2012.
[8] Hirschman I.I., *Infinite series*, Dover Publications, Reprint edition, October 15,2014.
[9] Ivić ,A., *Topics in recent zeta function theory*, Publications Mathématiques d'Orsay, Orsay, **1**, iii+272 (1983).
[10] Knopp, K. , *Theory and applications of infinite series*, Dover Publications, New York, 1990.
[11] Lewin L., *Polylogarithms and associated functions*, Elsevier, North Holland, 1981.
[12] Sîntămărian,A.,Furdui,O., *Sharpening Mathematical Analysis Skills*, Springer, 2021.
[13] Stojiljković ,V., Some series associated with central binomial coefficients and harmonic numbers, *Octogon Mathematical Magazine* , **29**(2), 749-763 (2021).
[14] Stojiljković, V.,Fabiano,N.,Šešum Čavić ,V., Harmonic series with polylogarithmic functions, *Vojnotehnički glasnik/Military Technical Courier*, **70**(1), 43-61 (2022).