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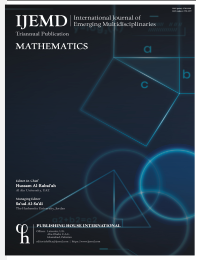

Current Developments on Extreme Value Copulas: Extended Pickands Dependence Functions

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Abstract

Copulas are mathematical tools used to model the dependence structure between random variables. Extreme value copulas specifically focus on capturing the tail dependence, which refers to the dependence structure between random variables when they exhibit extreme or rare events. The Pickands dependence functions are special convex functions that play a crucial role in characterizing extreme value copulas; they quantify the strength of their tail dependence. The creation of new Pickands dependence functions enhances our understanding of complex interdependencies, enabling more accurate modeling and risk assessment in diverse systems. In this article, a theoretical contribution to the topic is provided; an original strategy for generating new Pickands dependence functions based on existing ones is developed. The resulting Pickands dependence functions have the features of using the functionalities of standard functions (exponential, trigonometric, hyperbolic, etc.) and/or depend on several tuning parameters of various natures, which are quite uncommon in the literature. Two new extreme value copulas are derived from our findings. Their asymmetric and tail-dependent flexibility are emphasized. Numerical and graphical illustrations are given to support some theoretical facts.

Keywords: *Asymmetry; convexity; correlation measures; extreme value copulas; tail dependence.*
2010 Mathematics Subject Classification: *60E15; 62H99.*

1. Introduction

In order to understand the mathematical setting where the Pickands dependence functions (PiDFs) naturally appear, a presentation of the copulas is imposed. Basically, copulas are multivariate tools used to model the dependence structure between random variables. They provide a flexible framework for capturing various types of dependence, including tail dependence. The essential references about copulas are [1–5], and, for some recent developments, one can refer to [6–9].
For the purposes of this article, let us focus on the notion of tail dependence. Basically, it refers to the behavior of random variables in the upper or lower tails of their distributions. It describes whether extreme events occur simultaneously or whether there is a tendency for one variable to be extreme when the other is not. Tail dependence is crucial in various fields, including finance (see [10]), insurance (see [11]), and environmental science (see [12]), where understanding the joint behavior of extreme events is essential for risk assessment and management. Extreme value (EV) copulas, in particular, are specialized copulas that focus on modeling extreme events and tail dependence. They are derived from the theory of EV, which deals with the statistical behavior of extreme events. In terms of applications, the EV copulas find wide use in financial risk management. In particular, they are employed in the modeling of portfolio risk, credit risk, and the estimation of EV-at-risk and conditional value-at-risk. The EV copulas are also used in environmental science for analyzing extreme weather events, such as floods and hurricanes. See [5, 13–17].
One key ingredient in the mathematical definition of an EV copula is the PiDF as described in [18]. The PiDF provides a concise and interpretable representation of the tail dependence structure of the associated EV copula. Thus, it is naturally used to select the appropriate EV copulas that capture the desired tail dependence

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structure. With the EV copula methodology in mind, for a given PiDF denoted by $A(x)$, the corresponding EV copula is expressed as

$$C_{op}(x,y) = \exp\left\{\log(xy)A\left(\frac{\log(y)}{\log(xy)}\right)\right\}, \quad (x,y) \in [0,1]^2. \tag{1.1}$$

Basically, for $A(x) = 1$, the independence copula is obtained, i.e., $C_{op}(x,y) = xy$. In the general case, a PickDF can be presented as a convex function with a more or less peak bathtub shape that relies on the points $(0,1)$ and $(1,1)$ in the domain of the inverted triangle of $[0,1]^2$. Famous PiDFs include the Gumbel (Gu), Tawn (T), Marshall-Olkin (MO), Galambos (G), and Joe (J) PiDFs. See [5]. Some modern variants exist, as in [19–21], but attempts to create new PiDFs based on existing ones to extend their modeling capacities are rare. It is, however, of interest because the creation of fresh PiDFs enables more accurate and thorough modeling of tail dependence between random variables. Researchers can capture a wider range of tail dependence patterns and enhance the precision of modeling extreme events by investigating various functional forms.

This article proposes innovations in this setting. We elaborate a general strategy that is able to transform a known PiDF into a new PiDF. New functionalities are activated based on standard functions (exponential, trigonometric, hyperbolic, etc.), with the possible addition of several tuning parameters. A wide panel of different PiDFs can be created, beyond those always used by default. The resulting PiDFs are original in definition and demonstrate a high level of flexibility. These aspects are emphasized with complementary theoretical, graphical, and numerical facts. Two new EV copulas are derived from our findings. We study all their main properties, making them operational for future applications. On the other hand, our PiDFs can be considered independently for their convex nature only. Indeed, convex functions play a crucial role in optimization problems, ensuring efficient and reliable solutions. They also provide fundamental tools for proving theorems and establishing mathematical properties with wide-ranging applications. It is possible that our findings find use beyond the EV copula setting of the article.

The rest of the paper is structured as follows: Section 2 presents our general strategy, with several examples. A collection of parametric PiDFs is elaborated in Section 3. Section 4 concludes the article.

2. A general strategy of creation

Our strategy, consisting of generating PiDFs based on existing PiDFs, is described in this section. Simple examples are investigated.

2.1. On the notion of PiDF

From a mathematical viewpoint, a PiDF is an univariate function that satisfies precise conditions, which are described below (see [18]).

Definition 2.1. *A PiDF is an univariate function, say $A(x)$, $x \in [0,1]$, that satisfies the following conditions:*

Cond1 $A(0) = A(1) = 1$,
Cond2 $A(x) \geq \max(x, 1-x)$ *for any* $x \in [0,1]$,
Cond3 $A(x)$ *is convex.*

In other words, $A(x)$ is a convex function with a more or less peak bathtub shape that relies on the points $(0,1)$ and $(1,1)$ in the upper area of the triangle shape defined by $\max(1-x,x)$, $x \in [0,1]$. Some famous and recent examples of PiDFs are listed in Table 1.

Table 1: Main examples of PiDFs, with conditions and references.

Name	PiDF $A(x)$, $x \in [0,1]$	Condition	Reference
Gu	$[x^a + (1-x)^a]^{1/a}$	$a \geq 1$	[25]
T	$1-c+(c-b)x+[b^a x^a + c^a(1-x)^a]^{1/a}$	$a \geq 1, b \in [0,1], c \in [0,1]$	[27]
MO	$\max[1-b(1-x), 1-cx]$	$b \in [0,1], c \in [0,1]$	[26]
Ga	$1-[x^{-a}+(1-x)^{-a}]^{-1/a}$	$a > 0$	[24]
J	$1-[b^{-a}x^{-a}+c^{-a}(1-x)^{-a}]^{-1/a}$	$a > 0, b \in [0,1], c \in [0,1]$	[5]
BB5	$\{x^a+(1-x)^a-[x^{-ab}+(1-x)^{-ab}]^{-1/b}\}^{1/a}$	$a \geq 1, b > 0$	[5]
Ch1	$1+b\{[x^2+(1-x)\exp(-ax)]^{1/2}-1\}$	$a \in [0,1], b \in [0,1]$	[21]
Ch2	$[x^a+x^{a/2}(1-x)^{a/2}+(1-x)^a]^{1/a}$	$a \geq 2$	[21]

In some sense, the PiDFs are the univariate functional prints of the EV copulas. Based on a given PiDF $A(x)$, the associated EV copula $C_{op}(x,y)$ is defined as in Equation (1.1). The modeling of extreme events and tail dependence is the primary goal of EV copulas. Applications for the EV copulas are numerous, including finance, insurance, and environmental science. In light of this, the development of new PiDFs allows for more refined and comprehensive modeling of tail dependence between random variables. By exploring different functional forms, researchers can capture a wider range of tail dependence patterns and improve the accuracy of modeling extreme events. To this end, a general result is presented in the next subsection.

2.2. A general result

In the current state, the two classical ways to develop new PiDFs are: (i) making attempts to create functions $A(x)$ that fulfill the conditions in Definition 2.1, or (ii) using diverse convex combinations of existing PiDFs. A new way is presented in the next result, allowing us to define new PiDFs based on a composite transformation of existing PiDFs.

Proposition 2.2. Let $T(x)$, $x \in [0,1]$, be a function such that

CondA: $T(1) = 1$,
CondB: $T(x) \geq x$ for any $x \in [0,1]$,
CondC: $T(x)$ is non-decreasing,
CondD: $T(x)$ is convex.

Let $A(x)$, $x \in [0,1]$, be a PiDF as defined in Definition 2.1. Then the composite function

$$B(x) = T[A(x)], \quad x \in [0,1],$$

is a valid PiDF.

Proof. Let us prove that $B(x)$ satisfies the three conditions in Definition 2.1, i.e., Cond1, Cond2 and Cond3.

Proof of Cond1: Since $A(0) = A(1) = 1$ and $T(1) = 1$, we have $B(0) = T[A(0)] = T(1) = 1$ and $B(1) = T[A(1)] = T(1) = 1$.

Proof of Cond2: Since $T(x) \geq x$ and $A(x) \geq \max(x, 1-x)$ for any $x \in [0,1]$, we immediately have

$$B(x) = T[A(x)] \geq A(x) \geq \max(x, 1-x).$$

Proof of Cond3: Since $A(x)$ is convex and $T(x)$ is non-decreasing and convex, for any $\lambda \in [0,1]$ and $(x,y) \in [0,1]^2$, we have

$$\begin{aligned} B[\lambda x + (1-\lambda)y] &= T\{A[\lambda x + (1-\lambda)y]\} \leq T[\lambda A(x) + (1-\lambda)A(y)] \\ &\leq \lambda T[A(x)] + (1-\lambda)T[A(y)] = \lambda B(x) + (1-\lambda)B(y). \end{aligned}$$

As a result, $B(x)$ is convex.

This ends the proof. □

A simple example of function satisfying the conditions in Proposition 2.2 is $T(x) = x$, and it is the only one such that $T(0) = 0$. Another simple polynomial example is

$$T(x) = \frac{1}{2}(1+x^2), \quad x \in [0,1]. \tag{2.1}$$

Indeed, we have $T(1) = 1$, $T(x) \geq x$ since $(1-x)^2 \geq 0$, $T'(x) = x \geq 0$ implying that $T(x)$ is non-decreasing, and $T''(x) = 1 \geq 0$ implying that $T(x)$ is convex.
Based on the findings of Proposition 2.2 applied with the function $T(x)$ in Equation (2.1) and the PiDFs $A(x)$ in Table 1, some modified known PiDFs of the form $B(x) = T[A(x)]$ are listed in Table 2.

Table 2: Modified (Mo) PiDFs of the form $B(x) = T[A(x)]$ based on Proposition 2.2 with the function $T(x)$ in Equation (2.1) and the PiDFs $A(x)$ in Table 1.

Name	PiDF $B(x)$, $x \in [0,1]$	Condition
MoGu	$\frac{1}{2}\left\{1+[x^a+(1-x)^a]^{2/a}\right\}$	$a \geq 1$
MoT	$\frac{1}{2}\left(1+\left\{1-c+(c-b)x+[b^a x^a+c^a(1-x)^a]^{1/a}\right\}^2\right)$	$a \geq 1$, $b \in [0,1]$, $c \in [0,1]$
MoMO	$\frac{1}{2}\left(1+\{\max[1-b(1-x),1-cx]\}^2\right)$	$b \in [0,1]$, $c \in [0,1]$
MoGa	$\frac{1}{2}\left(1+\left\{1-[x^{-a}+(1-x)^{-a}]^{-1/a}\right\}^2\right)$	$a > 0$
MoJ	$\frac{1}{2}\left(1+\left\{1-[b^{-a}x^{-a}+c^{-a}(1-x)^{-a}]^{-1/a}\right\}^2\right)$	$a > 0$, $b \in [0,1]$, $c \in [0,1]$
MoBB5	$\frac{1}{2}\left(1+\{x^a+(1-x)^a-[x^{-ab}+(1-x)^{-ab}]^{-1/b}\}^{2/a}\right)$	$a \geq 1$, $b > 0$
MoCh1	$\frac{1}{2}\left[1+\left(1+b\left\{[x^2+(1-x)\exp(-ax)]^{1/2}-1\right\}\right)^2\right]$	$a \in [0,1]$, $b \in [0,1]$
MoCh2	$\frac{1}{2}\left\{1+[x^a+x^{a/2}(1-x)^{a/2}+(1-x)^a]^{2/a}\right\}$	$a \geq 2$

Each of the PiDFs presented in this table can be the subject of an independent study. In the next subsection, we focus on the MoT PiDF because it represents an original variant of the T PiDF with comparable asymmetric properties. Other examples with additional tuning parameters will be presented in Section 3.

2.3. Highlight of the MoT PiDF

With reference to Table 1, the MoT PiDF is defined as

$$B(x) = \frac{1}{2}\left(1+\left\{1-c+(c-b)x+[b^a x^a+c^a(1-x)^a]^{1/a}\right\}^2\right), \quad x \in [0,1].$$

If $A(x)$ denotes the T PiDF then, by construction, the following ordering holds: $B(x) \geq A(x)$ for any $x \in [0,1]$. Thus, they reach different modeling targets in this sense. When $b \neq c$, the MoT PiDF is clearly asymmetric; there exists $x \in [0,1]$ such that $B(x) \neq B(1-x)$. When $b = c$, it corresponds to a new variant of the MoG PiDF defined as

$$B(x) = \frac{1}{2}\left(1+\left[1+b\left\{[x^a+(1-x)^a]^{1/a}-1\right\}\right]^2\right), \quad x \in [0,1].$$

Figure 2.1 shows its shape for various values of a, b and c. In it, the following particular MO PickDF is displayed: $A(x) = \max(1-x,x)$, $x \in [0,1]$, because we know that a valid PiDF takes its values in its upper domain intersected with $[0,1]^2$. The software R is used for this figure and all the coming graphical and computational work (see [22]).

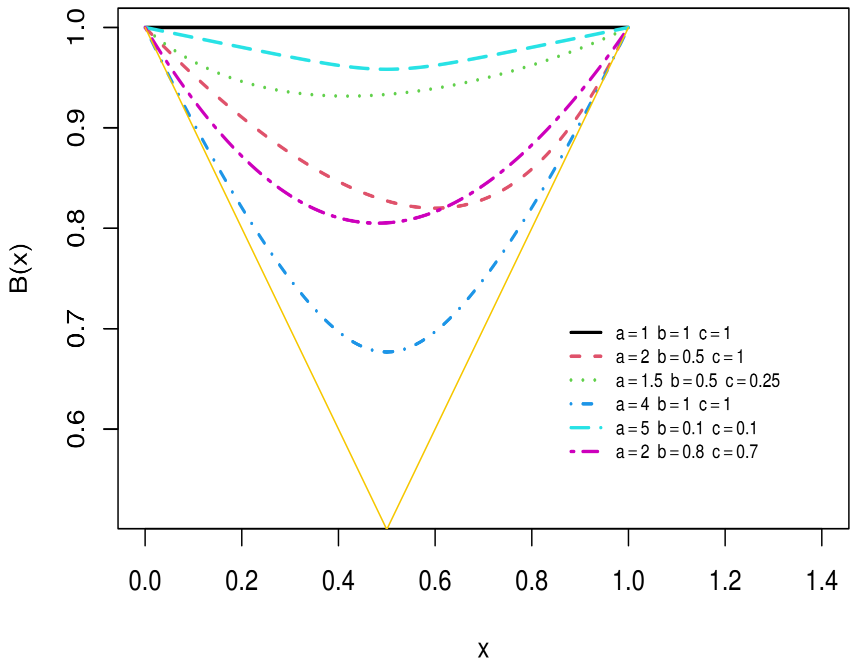

Figure 2.1: Plots of the MoT PiDF for various values of a, b and c.

From this figure, it is clear that $B(0) = B(1) = 1$, $B(x) \geq \max(x, 1-x)$ for any $x \in [0,1]$, and $B(x)$ is convex; the MoT PiDF is valid for the considered values of the parameters. Furthermore, it demonstrates a high level of flexibility; various symmetrical and asymmetrical curves are observed, and they can reach a large part of the domain of the inverted triangle.

In light of this, it is natural to investigate the corresponding EV copula. Based on Equation (1.1), it is indicated as

$$
\begin{aligned}
&C_{op}(x,y) = \exp\left\{\log(xy) B\left(\frac{\log(y)}{\log(xy)}\right)\right\} \\
&= \sqrt{xy}\exp\left(\frac{1}{2\log(xy)}\left\{(1-c)\log(x) + (1-b)\log(y) - \{b^a[-\log(y)]^a + c^a[-\log(x)]^a\}^{1/a}\right\}^2\right), \\
&\quad (x,y) \in [0,1]^2.
\end{aligned}
$$

Let us call it the MoT copula. We illustrate its shape in Figure 2.2 for various values of a, b and c.

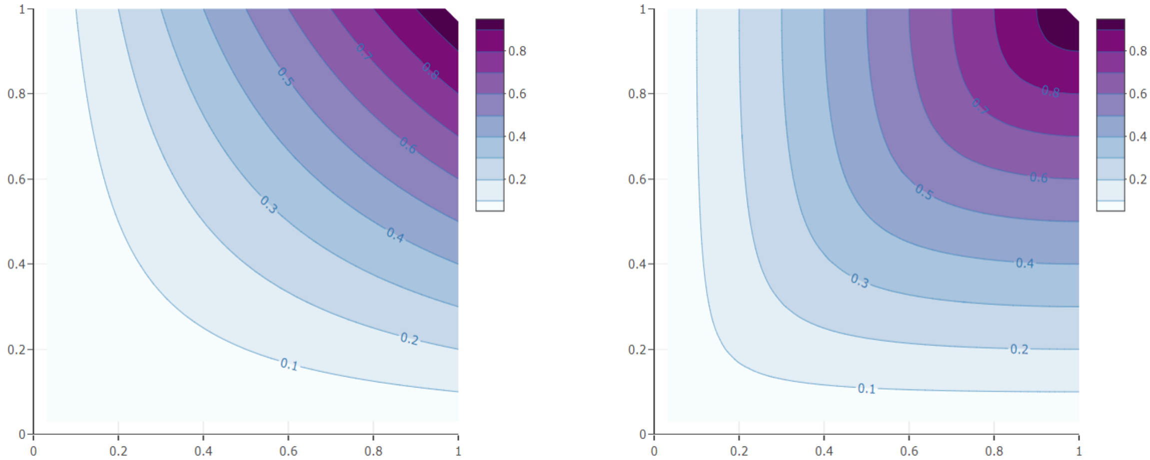

Figure 2.2: Contour-intensity plots of the MoT copula for $a = b = c = 1$ (left) and $a = 4$, and $b = c = 1$ (right).

This figure shows the validity of the MoT copula and also the effect of the parameters; a significant deformation is observed with the increasing value of a, revealing a certain degree of adaptability.
Clearly, for $b \neq c$ and $a \neq 1$, the MoT copula is not symmetric. It is not associative, implying that it is not Archimedean. Like for all the EV copulas, it satisfies the following power-geometric property: for any $(x,y) \in [0,1]^2$ and $s > 0$, we have

$$C_{op}(x,y) = [C_{op}(x^s,y^s)]^{1/s}.$$

Furthermore, the following inequalities hold: for any $(x,y) \in [0,1]^2$, we have

$$xy \leq C_{op}(x,y) \leq \min(x,y).$$

The left side of this inequality implies that the MoT copula is positively quadrant dependent. Since $B(1/2) > 1/2$, it has no lower tail dependence. Its upper tail dependence coefficient is given as

$$\lambda_{up} = 2\left[1 - B\left(\frac{1}{2}\right)\right] = 1 - \frac{1}{4}\left[2 - c - b + (b^a + c^a)^{1/a}\right]^2.$$

We can observe the complex influence of the parameters a, b, and c, on λ_{up}. Because of this, the MoT copula is particularly interesting for capturing upper tail dependence.
On the other hand, the Blomqvist coefficient is specified as

$$\beta_{Bl} = 2^{\lambda_{up}} - 1 = 2^{1-(1/4)\left[2-c-b+(b^a+c^a)^{1/a}\right]^2} - 1.$$

This measure of association has the advantage of being explicit in form. See [5] for more information. A numerical work is provided in Table 3; the values of β_{Bl} for various values of a, b and c are given.

Table 3: Sample of values for β_{Bl} for various values of a, b and c.

β_{Bl} | $a \rightarrow$	1.0	1.8	2.6	3.4	4.2	5.0	5.8	6.6	7.4
$b = 1, c = 1$	0	0.38	0.49	0.54	0.57	0.59	0.6	0.62	0.62
$b = 0.8, c = 0.9$	0	0.32	0.42	0.46	0.49	0.5	0.52	0.52	0.53
$b = 1, c = 0.2$	0	0.12	0.14	0.14	0.14	0.14	0.14	0.14	0.14
$b = 0.1, c = 0.6$	0	0.06	0.07	0.07	0.07	0.07	0.07	0.07	0.07

This table illustrates the numerical versatility of β_{Bl}, and the flexible positive dependence of the MoT copula. Indeed, we observe that $\beta_{Bl} \in [0, 0.62]$ for the considered values of a, b and c, with various increasing trends.

Another measure of association is the Spearman coefficient. With regard to our MoT copula defined with the MoT PiDF $B(x)$, it is expressed as

$$\rho_{Sp} = 12\int_0^1 \frac{1}{[1+B(x)]^2}dx - 3$$
$$= 12\int_0^1 \frac{1}{\left(3/2+(1/2)\left\{1-c+(c-b)x+[b^a x^a + c^a(1-x)^a]^{1/a}\right\}^2\right)^2}dx - 3.$$

See [5] for more information on the general formula. To our knowledge, the integral term cannot be calculated in an algebraic manner. For this reason, a numerical study is proposed. Table 4 presents the values of ρ_{Sp} for various values of a, b and c.

Table 4: Sample of values for ρ_{Sp} for various values of a, b and c.

ρ_{Sp} | $a \rightarrow$	1.0	1.8	2.6	3.4	4.2	5.0	5.8	6.6	7.4
$b=1, c=1$	0	0.54	0.67	0.72	0.74	0.75	0.76	0.76	0.77
$b=0.8, c=0.9$	0	0.46	0.57	0.61	0.63	0.64	0.64	0.65	0.65
$b=1, c=0.2$	0	0.19	0.23	0.24	0.25	0.25	0.25	0.25	0.25
$b=0.1, c=0.6$	0	0.1	0.12	0.12	0.13	0.13	0.13	0.13	0.13

From this table, for the considered values of a, b and c, we have $\rho_{Sp} \in [0, 0.77]$, which is quite acceptable for an EV copula.

To summarize this part, the MoT copula reveals itself to be a valuable EV copula with interesting asymmetric and dependence properties. It can be viewed as a natural alternative to the T copula for modeling the asymmetric structure of dependence when dealing with EVs. The MoT copula is thus recommended when the T copula demonstrates some limit to capturing the whole dependence structure, which can happen in fields such as finance, insurance, and environmental science.

As with any other copula, it can be used to generate new bivariate distributions with the composite scheme: $C_{op}[F(x), G(y)]$, $(x,y) \in \mathbb{R}^2$, where $F(x)$ and $G(y)$ denote two cumulative distribution functions. For statistical scenarios with positive values, which are often encountered in practice, see the review in [23] for suitable choices for these cumulative distribution functions.

The rest of the article is devoted to the creation of new parametric PiDFs, from which original EV copulas can be easily derived. A focus will be put on a new extension of the Gu PiDF and the Gu copula as well.

3. A collection of parametric PiDFs

This section is devoted to a special parametric transformation $T(x)$ based on power and integral terms that satisfies the conditions in Proposition 2.2. A collection of parametrically transformed PiDFs is derived.

3.1. A general transformation

The next result describes the proposed transformation.

Proposition 3.1. Let $(a,b,c) \in \mathbb{R}^3$. Let us consider the following integral function:

$$T(x) = \left[1 - \ell m \int_{-k(1-x)}^{0} f(t)dt\right]^{1/m}, \quad x \in [0,1], \tag{3.1}$$

where

- $k \in [0,1]$, $\ell \in [0,1]$ and $m \in (0,1]$
- $f(t)$, $t \in [-1,0]$, is a function such that

  CondI: $f(t) \in [0,1]$ for any $t \in [-1,0]$,

CondII: $f'(t) \geq 0$ for any $t \in [-1,0]$ (so $f(t)$ is non-decreasing on $[-1,0]$).

Then $T(x)$ satisfies the conditions in Proposition 2.2.

Proof. To begin, it is worth noting that $T(x)$ is well defined if the term under the exponent is positive, which is the case under the listed conditions. Indeed, for any $x \in [0,1]$, since $k \in [0,1]$, $\ell \in [0,1]$ and $m \in (0,1]$, it is clear that $\ell mk(1-x) \in [0,1]$, and the condition $f(t) \in [0,1]$ for any $t \in [-1,0]$, implies that

$$\ell m \int_{-k(1-x)}^{0} f(t)dt \leq \ell m \int_{-k(1-x)}^{0} dt = \ell mk(1-x) \leq 1.$$

Therefore, the exponentiated term has a mathematical sense, and $T(x)$ is well defined.
Subsequently, let us prove that $T(x)$ satisfies the four conditions in Proposition 2.2, i.e., CondA, CondB, CondC and CondD.

Proof of CondA: We have

$$T(1) = \left[1 - \ell m \int_{-k(1-1)}^{0} f(t)dt\right]^{1/m} = [1 - \ell m \times 0]^{1/m} = 1.$$

Proof of CondB: Since $f(t) \in [0,1]$ for any $t \in [-1,0]$, we have

$$\begin{aligned} T(x) &= \left[1 - \ell m \int_{-k(1-x)}^{0} f(t)dt\right]^{1/m} \geq \left[1 - \ell m \int_{-k(1-x)}^{0} dt\right]^{1/m} \\ &= [1 - \ell mk(1-x)]^{1/m}. \end{aligned}$$

For any $x \in [0,1]$, since $k \in [0,1]$, $\ell \in [0,1]$ and $m \in (0,1]$, it is clear that $\ell mk(1-x) \in [0,1]$. Therefore, by applying the Bernoulli inequality, i.e., $(1+x)^r \geq 1+rx$, with $x \geq -1$ and $r \geq 1$, with the exponent $r = 1/m \geq 1$, and using $\ell k \in [0,1]$, we get

$$T(x) \geq 1 - \ell mk\frac{1}{m}(1-x) = 1 - \ell k(1-x) \geq 1 - (1-x) = x.$$

The desired result is established.
Proof of CondC: By applying standard differentiation rules, we obtain

$$T'(x) = \ell kf[-k(1-x)]\left[1 - \ell m \int_{-k(1-x)}^{0} f(t)dt\right]^{1/m-1}. \tag{3.2}$$

Since $\ell k \in [0,1]$ and $f(t) \in [0,1]$ for any $t \in [-1,0]$, it is clear that $f[-k(1-x)] \geq 0$, and so $T'(x) \geq 0$. Hence $T(x)$ is non-decreasing.
Proof of CondD: By applying standard differentiation rules based on Equation (3.2), we obtain

$$\begin{aligned} T''(x) &= \ell k^2 f'[-k(1-x)]\left[1 - \ell m \int_{-k(1-x)}^{0} f(t)dt\right]^{1/m-1} \\ &+ m\left(\frac{1}{m} - 1\right)\ell^2 k^2 \{f[-k(1-x)]\}^2 \left[1 - \ell m \int_{-k(1-x)}^{0} f(t)dt\right]^{1/m-2}. \end{aligned}$$

Since $f'(t) \geq 0$ for any $t \in [-1,0]$, it is clear that $f'[-k(1-x)] \geq 0$ and, since $m \in (0,1]$, we have $1/m - 1 \geq 0$. All the rest of the main terms are non-negative (with the same arguments as those of the previous point), we conclude that $T''(x) \geq 0$, meaning that $T(x)$ is convex.

This ends the proof. □

This result shows how Proposition 2.2 can be applied to generate sophisticated PiDFs involving diverse kinds of functions with several parameters. The next section illustrates this claim by presenting a collection of six new PiDFs.

3.2. A collection of PiDFs

Some illustrative examples of Proposition 2.2 are offered below.

Lemma 3.2. *The following functions $f(t)$, $t \in [-1,0]$, satisfy the conditions CondI and CondII in Proposition 3.1.*

Func1: $f(t) = (1+t)^n$ *with* $n \geq 0$,
Func2: $f(t) = \exp[-(-t)^r]$ *where r is a non-negative integer,*
Func3: $f(t) = [\mathrm{sech}(t)]^n$ *with* $n \geq 0$,
Func4: $f(t) = \{\mathrm{sinc}[(\pi/2)t]\}^n$ *with* $n \geq 0$,
Func5: $f(t) = \{2/[1+\exp(-t)]\}^n$ *with* $n \geq 0$,
Func6: $f(t) = \{\cos[(\pi/2)t]\}^n$ *with* $n \geq 0$.

Proof. Let us verify succinctly the conditions CondI and CondII for each function.

Proof for Func1: For any $t \in [-1,0]$ and $n \geq 0$, it is clear that $f(t) = (1+t)^n \in [0,1]$; CondI is satisfied. Furthermore, we have $f'(t) = n(1+t)^{n-1} \geq 0$; CondII is satisfied.

Proof for Func2: For any $t \in [-1,0]$ and non-negative integer r, since $(-t)^r \in [0,1]$, we have $f(t) = \exp[-(-t)^r] \in [0,1]$; CondI is satisfied. Furthermore, we have $f'(t) = r(-t)^{r-1}\exp[-(-t)^r] \geq 0$; CondII is satisfied.

Proof for Func3: For any $t \in [-1,0]$ and $n > 0$, since $\mathrm{sech}(t) = 2/[\exp(t)+\exp(-t)] \in [0,1]$, it is immediate that $f(t) = [\mathrm{sech}(t)]^n \in [0,1]$; CondI is satisfied. Furthermore, we have $f'(t) = n\sinh(-t)[\mathrm{sech}(t)]^{n+1} \geq 0$; CondII is satisfied.

Proof for Func4: For any $t \in [-1,0]$ and $n > 0$, since $\mathrm{sinc}[(\pi/2)t] \in [0,1]$, we have $f(t) = \{\mathrm{sinc}[(\pi/2)t]\}^n \in [0,1]$; CondI is satisfied. Furthermore, we have

$$f'(t) = n\frac{\pi t\cos[(\pi/2)t] - 2\sin[(\pi/2)t]}{\pi t^2}\{\mathrm{sinc}[(\pi/2)t]\}^{n-1}.$$

The following inequality is well-known: $\sin(x) \geq x\cos(x)$ for $x \in [0,\pi/2]$. By applying it with $x = -(\pi/2)t$ (and $t \in [-1,0]$), we obtain $\pi t\cos[(\pi/2)t] - 2\sin[(\pi/2)t] \geq 0$, which implies that $f'(t) \geq 0$; CondII is satisfied.

Proof for Func5: For any $t \in [-1,0]$, since $1+\exp(-t) \geq 2$, it is immediate that $f(t) = \{2/[1+\exp(-t)]\}^n \in [0,1]$; CondI is satisfied. Furthermore, we have

$$f'(t) = n\frac{2\exp(-t)}{[1+\exp(-t)]^2}\left[\frac{2}{1+\exp(-t)}\right]^{n-1} \geq 0.$$

Hence, CondII is satisfied.

Proof for Func6: For any $t \in [-1,0]$, it is immediate that $f(t) = \{\cos[(\pi/2)t]\}^n \in [0,1]$; CondI is satisfied. Furthermore, we have $f'(t) = n(\pi/2)\sin[-(\pi/2)t]\{\cos[(\pi/2)t]\}^{n-1} \geq 0$. Hence, CondII is satisfied.

This ends the proof. □

Based on Proposition 3.1 and Lemma 3.2, Table 5 presents some parametric functions $T(x)$ as given in Equation (3.1). For Func4, we have considered the sine integral function defined by $\mathrm{Si}(x) = \int_0^x [\sin(t)/t]dt$.

Table 5: Transformed PiDFs based on Proposition 2.2 with the function $T(x)$ in Equation (3.1); $\mathscr{S}$ represents the conditions $k \in [0,1]$, $\ell \in [0,1]$ and $m \in (0,1]$.

Function $f(t)$	function $T(x)$	Condition
Func1	$\left(1 - \ell m\frac{1}{n+1}\left\{1-[1-k(1-x)]^{n+1}\right\}\right)^{1/m}$	$\mathscr{S}$, $n \geq 0$
Func2, $r=1$	$[1-\ell m\{1-\exp[-k(1-x)]\}]^{1/m}$	$\mathscr{S}$
Func3, $n=2$	$\{1-\ell m\tanh[k(1-x)]\}^{1/m}$	$\mathscr{S}$
Func4, $n=1$	$\left\{1-\ell m\frac{2}{\pi}\mathrm{Si}\left[\frac{\pi}{2}k(1-x)\right]\right\}^{1/m}$	$\mathscr{S}$
Func5, $n=1$	$\left[1-2\ell m\log\left\{\frac{2}{1+\exp[-k(1-x)]}\right\}\right]^{1/m}$	$\mathscr{S}$
Func6, $n=1$	$\left\{1-\ell m\frac{2}{\pi}\sin\left[\frac{\pi}{2}k(1-x)\right]\right\}^{1/m}$	$\mathscr{S}$

If we consider a PiDF $A(x)$ and one of the functions $T(x)$ in Table 5, then Proposition 2.2 guarantees that $B(x) = T[A(x)]$ is a valid PiDF. For instance, we can consider the Gu PiDF defined as $A(x) = [x^a + (1-x)^a]^{1/a}$ and the first function $T(x)$ in Table 5 with $n = 0$ and $\ell = 1$ to simplify the situation. This gives the following three-parameter PiDF:

$$B(x) = T[A(x)] = \left(1 - mk\left\{1 - [x^a + (1-x)^a]^{1/a}\right\}\right)^{1/m}, \quad x \in [0,1],$$

with $a \geq 1$, $m \in (0,1]$ and $k \in [0,1]$. More examples are provided in Table 6, with the functions $T(x)$ in Table 5 and still with the Gu PiDF for $A(x)$.

Table 6: A collection of transformed (Tr) PiDFs based on $B(x) = T[A(x)]$, with the functions $T(x)$ in Table 5 and the Gu PiDF $A(x)$; $\mathscr{U}$ represents the conditions $a \geq 1$, $k \in [0,1]$, $\ell \in [0,1]$ and $m \in (0,1]$.

Name	PiDF $B(x)$	Condition
TrGu1	$\left(1 - \ell m \frac{1}{n+1}\left\{1 - [1 - k(1 - [x^a + (1-x)^a]^{1/a})]^{n+1}\right\}\right)^{1/m}$	$\mathscr{U}, n \geq 0$
TrGu2	$\left[1 - \ell m\left\{1 - \exp[-k(1 - [x^a + (1-x)^a]^{1/a})]\right\}\right]^{1/m}$	$\mathscr{U}$
TrGu3	$\left\{1 - \ell m \tanh[k(1 - [x^a + (1-x)^a]^{1/a})]\right\}^{1/m}$	$\mathscr{U}$
TrGu4	$\left\{1 - \ell m \frac{2}{\pi} \operatorname{Si}\left[\frac{\pi}{2} k(1 - [x^a + (1-x)^a]^{1/a})\right]\right\}^{1/m}$	$\mathscr{U}$
TrGu5	$\left[1 - 2\ell m \log\left\{\frac{2}{1 + \exp[-k(1 - [x^a + (1-x)^a]^{1/a})]}\right\}\right]^{1/m}$	$\mathscr{U}$
TrGu6	$\left\{1 - \ell m \frac{2}{\pi} \sin\left[\frac{\pi}{2} k(1 - [x^a + (1-x)^a]^{1/a})\right]\right\}^{1/m}$	$\mathscr{U}$

All the PiDFs presented in this table are new. This demonstrates that we can construct a wild panel of PiDFs beyond the standard ones, i.e., those in Table 1. In particular, logarithmic, trigonometric, and hyperbolic functions can be involved.

3.3. Focus on a special TrGu1 PiDF

In order to highlight the importance of our findings, we focus on a special TrGu1 (STrGu1) PiDF, defined as the TrGu1 PiDF in Table 6 with $\ell = 1$ and $n = 0$ to simplify the situation. Thus, it is defined by

$$B(x) = \left(1 - mk\left\{1 - [x^a + (1-x)^a]^{1/a}\right\}\right)^{1/m}, \quad x \in [0,1],$$

with $a \geq 1$, $m \in (0,1]$ and $k \in [0,1]$. It is a new three-parameter PiDF. If $A(x)$ denotes the Gu PiDF then, by construction, the following ordering holds: $B(x) \geq A(x)$ for any $x \in [0,1]$. Thus, the STrGu1 and Gu PiDFs have a different ordering in this sense; they reach their own modeling objective. For any $x \in [0,1]$, it is clear that $B(x) = B(1-x)$; the STrGu1 PiDF is symmetric. When $k = m = 1$, it corresponds to the Gu PiDF.

As a visual approach, Figure 3.1 shows the shape of the STrGu1 PiDF for various values of a, k and m. Again, the special MO PickDF, i.e., $A(x) = \max(1-x, x)$, $x \in [0,1]$, is displayed as a PiDF of reference.

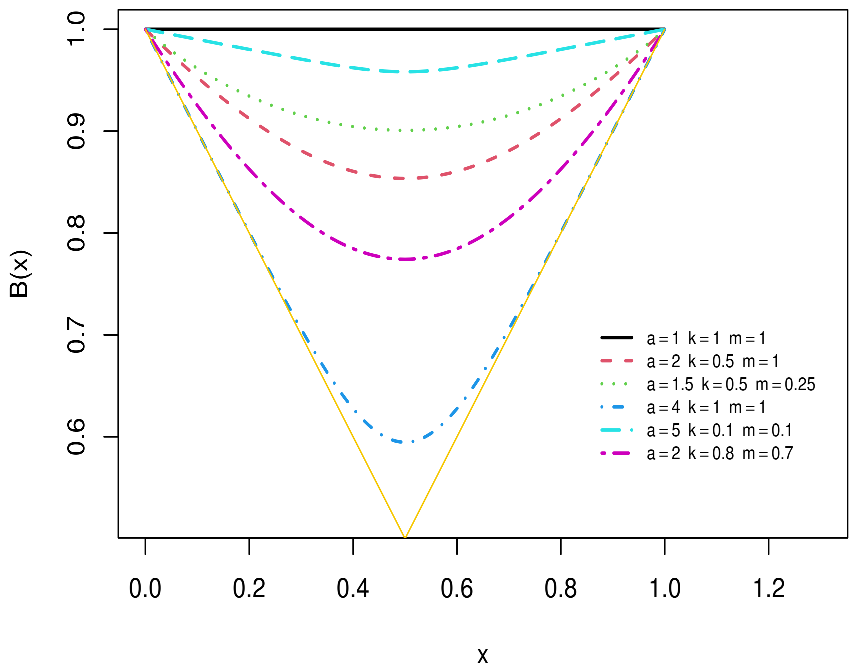

Figure 3.1: Plots of the STrGu1 PiDF for various values of a, k and m.

We see that all the zones of the inverted triangle shape are reached, including the peak zone, but with symmetric curves only. This flexibility motivates us to investigate the associated EV copula. Based on Equation (1.1), it is expressed as

$$
\begin{aligned}
C_{op}(x,y) &= \exp\left\{\log(xy)B\left(\frac{\log(y)}{\log(xy)}\right)\right\} \\
&= \exp\left(\log(xy)\left[1-mk\left\{1+\frac{1}{\log(xy)}\{[-\log(y)]^a+[-\log(x)]^a\}^{1/a}\right\}\right]^{1/m}\right), \\
&\quad (x,y)\in[0,1]^2.
\end{aligned}
$$

Let us call it the STrGu1 copula. It is very versatile in terms of dependence structure, as illustrated in Figure 3.2.

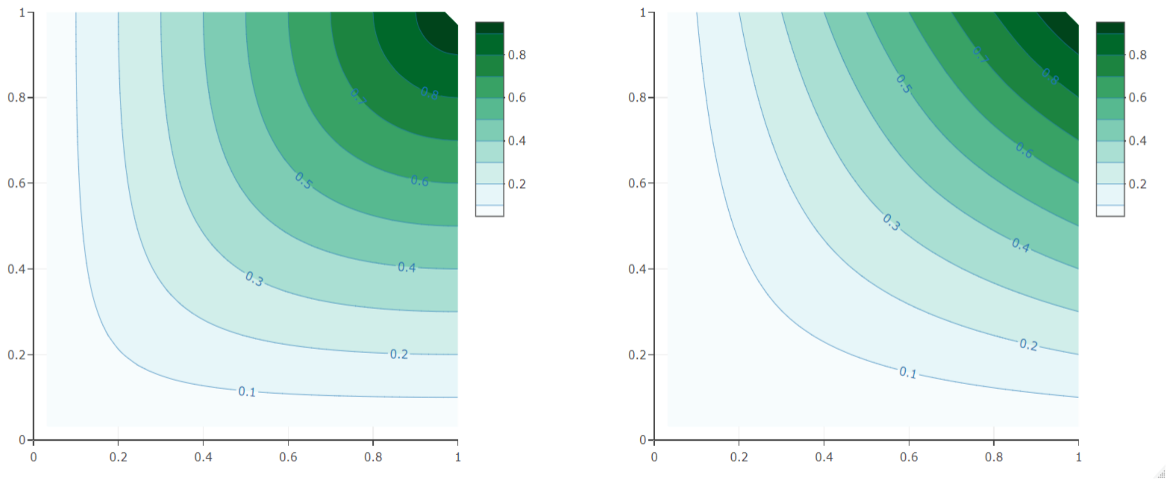

Figure 3.2: Contour-intensity plots of the STrGu1 copula for $a = 2$, $k = 1$, $m = 0.5$ (left) and $a = 4$, and $a = 4$, $k = 0.1$, $m = 1$ (right).

The validity of the STrGu1 copula is observed, and different contours and intensity levels are observed with the change of values of the parameters.

The STrGu1 copula is symmetric. It is not associative, implying that it is not Archimedean. Since it is an EV copula, the following power-geometric property is fulfilled: $C_{op}(x,y) = [C_{op}(x^s,y^s)]^{1/s}$ for any $(x,y) \in [0,1]^2$ and $s > 0$, and the following inequalities hold: $xy \leq C_{op}(x,y) \leq \min(x,y)$, implying that it is positively quadrant dependent. It has no lower tail dependence, except in the very special case where $B(1/2) = 1/2$, i.e., $\left[1 - mk\left(1 - 2^{1/a-1}\right)\right]^{1/m} = 1/2$, and its upper tail dependence coefficient is given as

$$\lambda_{up} = 2\left[1 - B\left(\frac{1}{2}\right)\right] = 2\left\{1 - \left[1 - mk\left(1 - 2^{1/a-1}\right)\right]^{1/m}\right\}.$$

We see how significant the effect of the parameters is on λ_{up}, making the STrGu1 copula of particular interest to capture diverse kinds of symmetric and upper tail dependence.

On the other hand, the Blomqvist coefficient is specified as

$$\beta_{Bl} = 2^{\lambda_{up}} - 1 = 2^{2\left\{1-\left[1-mk\left(1-2^{1/a-1}\right)\right]^{1/m}\right\}} - 1.$$

It can be easily manipulated from the mathematical and practical viewpoints. Table 7 collects a sample of numerical values of β_{Bl} for various values of a, k and m.

Table 7: Sample of values for β_{Bl} for various values of a, k and m.

β_{Bl} | $a \rightarrow$	1.0	1.8	2.6	3.4	4.2	5.0	5.8	6.6	7.4
$k = 1, m = 1$	0	0.44	0.62	0.71	0.77	0.8	0.83	0.85	0.87
$k = 0.8, m = 0.9$	0	0.34	0.46	0.52	0.56	0.59	0.61	0.62	0.63
$k = 1, m = 0.2$	0	0.39	0.52	0.58	0.62	0.65	0.66	0.68	0.69
$k = 0.1, m = 0.6$	0	0.04	0.05	0.05	0.06	0.06	0.06	0.06	0.06

For the considered values of the parameters, we see that $\beta_{Bl} \in [0, 0.87]$. This range is quite wide, the maximum for the positive values being $[0,1]$.

The Spearman coefficient is defined as

$$\rho_{Sp} = 12\int_0^1 \frac{1}{[1+B(x)]^2}dx - 3$$
$$= 12\int_0^1 \frac{1}{\left[1+\left(1-mk\left\{1-[x^a+(1-x)^a]^{1/a}\right\}\right)^{1/m}\right]^2}dx - 3.$$

Clearly, it is not possible to calculate the integral term in a standard way; a computational study is necessary. To this end, Table 8 presents a sample of numerical values of ρ_{Sp} for various values of a, k and m.

Table 8: Sample of values for ρ_{Sp} for various values of a, k and m.

ρ_{Sp} | $a \rightarrow$	1.0	1.8	2.6	3.4	4.2	5.0	5.8	6.6	7.4
$k=1, m=1$	0	0.62	0.8	0.88	0.92	0.94	0.96	0.97	0.9
$k=0.8, m=0.9$	0	0.47	0.6	0.66	0.68	0.7	0.71	0.71	0.72
$k=1, m=0.2$	0	0.56	0.7	0.76	0.78	0.8	0.81	0.81	0.82
$k=0.1, m=0.6$	0	0.05	0.07	0.07	0.07	0.07	0.07	0.07	0.08

For the considered values of the parameters, ρ_{Sp} almost reaches the perfect interval $[0,1]$. As a result, the STrGu1 copula is particularly adapted to model a wide panel of dependence structures. When dealing with an EV setting, this is particularly demanded in finance, insurance, and environmental science.

Remark 3.3. *Among other things, the introduced PiDFs are also new convex functions and can be used as such. Indeed, convex functions are not only essential in copula theory, but they also play a crucial role in areas such as economics, finance, engineering, and computer science. Developing new convex functions expands our mathematical toolkit, allowing us to tackle complex problems, provide efficient algorithms, and gain deeper insights into the underlying structures of various phenomena. We thus modestly contribute in this direction.*

4. Conclusion

Creating EV copulas and new PiDFs holds modern importance in the fields of statistics and risk management. These advanced techniques enable us to model and understand the dependence structures in extreme events, such as financial crises or natural disasters. By accurately capturing tail dependence, we can assess and manage risks more effectively, leading to improved decision-making and robust risk mitigation strategies in today's complex and interconnected world. In this article, a new mathematical strategy is developed to generate PiDFs based on existing ones. The well-known panel of PiDFs is thus significantly enlarged, offering original alternative options. Some of the new PiDFs are particularly able to adapt to various dependence analysis scenarios thanks to the presence of comprehensive tuning parameters. The importance of the findings is emphasized by exhibiting numerous examples. Some of the associated copulas were described and studied in detail with the help of graphics and numerical tables.
The natural perspective of this theoretical article is the application of the findings to real-world data, especially in finance, insurance, and environmental science, following the spirit of [10–12]. This is planned for future work.

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