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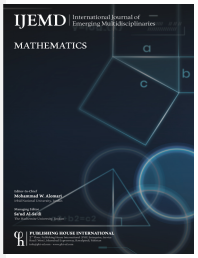

# On Inequalities of Trapezium Type Via Fractional Integrals Operators

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**Abstract**

In this article, we get solutions of some integral inequalities of Hermite-Hadamard type and using the approach of $(\psi,h)$-Convex function by the way of Riemann-Liouville Fractional integrals and Katugampola Fractional integral operators.

***Keywords:*** *Hermite-Hadamard inequality; Riemann-Liouville Fractional integrals; Katugampola Fractional integrals; $(\psi,h)$-Convex function*

***2010 Mathematics Subject Classification:*** *26D10; 90C26; 26A33.*

## 1. Introduction

The concept of convexity has played a dominant role and has gotten special attention by many researchers in various places of pure and applied sciences. It is noticed that the convex (concave) function is one of the most significant function which is furthermore generalized day by day see the References [5, 7, 8, 9, 13, 15]. One of the significant generalization from these references is $\psi$-Convex function which is furthermore generalized by using the concept of Raina's function as $(\psi,h)$-Convex function in [13] by R. Saima.
In literature, there are so many results related with convex or generalized convex function in inequalities, one of the popular inequality is Hermite-Hadamard inequality, which is widely seen in the mathematical literature.
The concept of $(\psi,h)$-Convexity provides a powerful tools in proving a large scale of inequalities. Several generalizations and extensions concerning to the below inequality have been proved by many researchers see the Reference [2] Dragomir and Agarwal.
Let g be a convex function on the finite interval $[v_1, v_2]$, then

$$g\left(\frac{v_1+v_2}{2}\right) \leq \frac{1}{v_2-v_1}\int_{v_1}^{v_2} g(x)dx \leq \frac{g(v_1)+g(v_2)}{2}$$

Different results have been established using this integral inequality by connecting it with Riemann-Liouville fractional integrals see for instance [4]. The above inequality has never ceased to fascinate researchers, several variants, extensions, generalization and improvements have been set up.
In [2] Dragomir and Agarwal derived the following Hermite-Hadamard type inequality in this form

$$\mid g\left(\frac{v_1+v_2}{2}\right) - \frac{1}{v_2-v_1}\int_{v_1}^{v_2} g(x)dx \mid \leq \frac{v_2-v_1}{8}\left(\frac{\mid g\prime(v_1)\mid + \mid g\prime(v_2)\mid}{2}\right)$$

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In [7] Kermausuor and Nwaeze proved the following inequality as

$$\frac{1}{v_2 - v_1}\int_{v_1}^{v_2} g(x)dx \leq \max[g(v_1), g(v_2)]$$

by connecting it with Katugampola fractional integral for strongly $\eta-$ quasi-convex function.
Different researchers has worked on different integral inequalities such as Hermite and Hadamard type, Simpson and Fejer type inequalitites, using the approach of generalizations of convex function such as $\eta-$convex, Quasi-convex, $(\eta,\psi)-$ *convex* and preinvex functions. For recent references see [1, 4, 5, 6, 7, 9, 10, 13].
The integral inequalities having different fractional integral operators such as Riemann-Liouville, katugampola and k-integral operator which have been considered in [4, 7, 14] respectively. A Since work in this direction has received many more attention, we try to introduce some useful formulations in this article such as the known results involving different fractional integrals and fractional integral operators become more generalized and comprehensible and these results give some new ideas to the upcoming researchers.
This section contains different basic definitions of convexity and operators,as well as some useful results that will be necessary for the development of the present work.

**Definition 1.1.** *[13] Let $\Omega \subseteq \mathbb{R}^n$ and a mapping $g : \Omega \to \mathbb{R}^n$ is said to be classical $\psi$-Convex function, if*

$$g(v_1 + \zeta F_{\varsigma}^{\vartheta,\eta}(v_2 - v_1)) \leq (1-\zeta)g(v_1) + \zeta g(v_2)\ \forall v_1, v_2 \in \Omega; \zeta \in [0,1]$$

*where $F_{\varsigma}^{\vartheta,\eta}$ denotes Raina's function which is defined in [13] as:*

$$F_{\varsigma}^{\vartheta,\eta}(z) = \sum_{k=0}^{\infty} \frac{\vartheta(k)}{\Gamma(k\varsigma+\eta)} z^k,$$

*Where $\varsigma, \eta > 0$ ; $|z| < \Re$ and $\vartheta = (\vartheta(0), ....., \vartheta(K), .....)$ be a bounded sequences of positive real numbers. If we take $\varsigma = 1, \eta = 0$, then*

$$\vartheta(k) = \frac{(\alpha_1)_k(\alpha_2)_k}{(\alpha_3)_k}, k = 0, 1, 2, .....,$$

*Where $\alpha_1, \alpha_2, \alpha_3$ are parameters which can take real and complex number provided ($\alpha_3 \neq 0, -1, -2, ...$) also,*

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)(a+2)....(a+k-1), k = 0, 1, 2, .....$$

*and restrict its domain to $|z| \leq 1$ with $z \in \mathbb{C}$ then we have a well known function known as hypergeometric function which is defined as*

$$F_{\varsigma}^{\vartheta,\eta}(z) = F(\alpha_1, \alpha_2, \alpha_3; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k}{(\alpha_3)_k} z^k$$

The concept of ($\psi$,$h$)-Convex function has been generalized by R. Saima in [13] as follows.

**Definition 1.2.** *[13] Let a non negative function be $h : (0,1) \subseteq \tau \to \mathbb{R}$ and $\Omega$ be a $\psi$-convex set. A function $g : \Omega \to \mathbb{R}^n$ is said to be ($\psi$,$h$)-Convex function, if*

$$g(v_1 + \zeta F_{\varsigma}^{\vartheta,\eta}(v_2 - v_1)) \leq h(1-\zeta)g(v_1) + h(\zeta)g(v_2)\ \forall v_1, v_2 \in \Omega; \zeta \in [0,1]$$

*where $F_{\varsigma}^{\vartheta,\eta}$ denotes Raina's function and $\vartheta = (\vartheta(0), ....., \vartheta(K), .....)$ be a bounded sequences of positive real numbers with $\varsigma, \eta > 0$*

**Remark 1.3.** *If $h(\zeta) = \zeta$ with $F_{\varsigma}^{\vartheta,\eta}(v_2 - v_1) = v_2 - v_1$ in Defination 2 then we get classical convex function.*

**Definition 1.4.** *[7] The left and right-sided Riemann-liouville fractional integrals of order $\alpha > 0$ of g are defined by*

$$ {}_kJ^{\alpha}_{v_1^+} g(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{v_1}^{x} (x-t)^{\frac{\alpha}{k}-1} g(t)dt, x > v_1 $$

$$ {}_kJ^{\alpha}_{v_2^-} g(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{x}^{v_2} (t-x)^{\frac{\alpha}{k}-1} g(t)dt, x < v_2 $$

*where k is positive and $\Gamma_k$ is the k-Gamma function defined by*

$$ \Gamma_k(x) = \int_0^{\infty} t^{x-1} e^{-\frac{t^k}{k}} dt $$

*Noted that $\Gamma_k(x+k) = x\Gamma_k(x)$*

**Definition 1.5.** *[7] let $[v_1, v_2] \subseteq \mathbb{R}$ be a finite interval .Then the left and right-sided Katugampola fractional integrals of order $\alpha > 0$ of $g \in X_c^p(v_1, v_2)$ are defined by*

$$ {}^{\rho}I^{\alpha}_{v_1^+} g(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{v_1}^{x} \frac{t^{\rho-1}}{(x^{\rho}-t^{\rho})^{1-\alpha}} g(t)dt $$

$$ {}^{\rho}I^{\alpha}_{v_2^-} g(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{x}^{v_2} \frac{t^{\rho-1}}{(x^{\rho}-t^{\rho})^{1-\alpha}} g(t)dt $$

*with $v_1 < x < v_2$ ;$\rho > 0$*

**Definition 1.6.** *[7] The left and right-sided Hadamard fractional integrals of order $\alpha > 0$ of g are defined by*

$$ H^{\alpha}_{v_1^+} g(x) = \frac{1}{\Gamma(\alpha)} \int_{v_1}^{x} \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{g(t)}{t} dt $$

*and*

$$ H^{\alpha}_{v_2^-} g(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{v_2} \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{g(t)}{t} dt $$

**Theorem 1.7.** *[6] Let $\alpha > 0$and $\rho > 0$ . Then for $x > a$*

$$ \lim_{\rho \to 1} {}^{\rho}I^{\alpha}_{v_1^+} g(x) = J^{\alpha}_{v_1^+} g(x) $$

$$ \lim_{\rho \to 0} {}^{\rho}I^{\alpha}_{v_1^+} g(x) = H^{\alpha}_{v_1^+} g(x) $$

*Where I, J, H are the Katugampola, Riemann-liouville and Hadamard fractional integral operators respectively. Similarly, these results hold for right-sided operators.*

**Theorem 1.8.** *For $\rho = k = 1$ .Then for $\alpha > 0$*

$$ {}_kJ^{\alpha}_{v_1^+} g(x) = {}^{\rho}I^{\alpha}_{v_1^+} g(x) $$

$$ {}_kJ^{\alpha}_{v_2^-} g(x) = {}^{\rho}I^{\alpha}_{v_2^-} g(x) $$

*Where I and J are the Katugampola and Riemann-liouville fractional integral operators respectively.*

**Theorem 1.9.** *[1] Let $\alpha, \rho > 0$ with a bounded sequence of real numbers $\vartheta = (\vartheta(0), ....., \vartheta(K), .....)$.Let $g : [v_1^{\rho}, v_2^{\rho}] \to \mathbb{R}$ be a $(\psi, h)$-Convex differentiable function on $(v_1^{\rho}, v_2^{\rho})$ with $0 \le v_1 \le v_2$. If the fractional integrals exists then*

$$ \frac{g(v_1^{\rho}) + g(v_2^{\rho})}{2} - \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(v_2^{\rho}-v_1^{\rho})^{\alpha}} [{}^{\rho}I^{\alpha}_{v_1^+} g(v_2^{\rho}) + {}^{\rho}I^{\alpha}_{v_2^-} g(v_1^{\rho})] $$

$$ = \frac{(v_2^{\rho}-v_1^{\rho})\rho}{2} \int_0^1 [1 - (\zeta F^{\vartheta}_{\varsigma,\eta})^{\rho})^{\alpha} - (\zeta F^{\vartheta}_{\varsigma,\eta})^{\rho\alpha}] $$

$$ \times g\prime((\zeta F^{\vartheta}_{\varsigma,\eta})^{\rho} v_1^{\rho} + (1 - (\zeta F^{\vartheta}_{\varsigma,\eta})^{\rho}))v_2^{\rho}))d\zeta, \varsigma, \eta > 0 $$

*Where I is Katugampola fractional integral operator and $F$ is Raina's function.*

In particular, this article have some generalizations about the Hermite-Hadamard inequalities using the approach of ($\psi$,$h$)-Convex function and fractional integral operators.

## 2. Main Results

**Theorem 2.1.** *Let $\alpha,k>0$ with a bounded sequence of real numbers $\vartheta=(\vartheta(0),.....,\vartheta(K),.....)$ and let $g:[v_1,v_2]\to\mathbb{R}$ be a $(\psi,h)$-Convex function then*

$$\frac{k\Gamma_k(\alpha)}{2((v_2-v_1)F_{\varsigma}^{\vartheta})^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{(v_1+(v_2-v_1)F_{\varsigma}^{\vartheta})^-}g(v_1)+{}_kJ^{\alpha}_{(v_2-(v_2-v_1)F_{\varsigma}^{\vartheta})^+}g(v_2)\right]$$
$$\leq\frac{g(v_1)+g(v_2)}{2}\int_0^1\zeta^{\frac{\alpha}{k}-1}(h(\zeta)+h(1-\zeta))d\zeta,\varsigma,\eta>0$$

*Where J is Riemann-liouville fractional integral operator and F is Raina's function.*

*Proof.*

$$g\left((1-\zeta F_{\varsigma}^{\vartheta},\eta)v_1+(\zeta F_{\varsigma}^{\vartheta},\eta)v_2\right)\leq h(1-\zeta)g(v_1)+h(\zeta)g(v_2) \tag{2.1}$$

$$g\left((\zeta F_{\varsigma}^{\vartheta},\eta)v_1+(1-\zeta F_{\varsigma}^{\vartheta},\eta)v_2\right)\leq h(\zeta)g(v_1)+h(1-\zeta)g(v_2) \tag{2.2}$$

Adding (2.1) and (2.2), multiply both sides by $\zeta^{\frac{\alpha}{k}-1}$ and then integrate from 0 to 1 we get

$$\int_0^1\zeta^{\frac{\alpha}{k}-1}g((1-\zeta F_{\varsigma}^{\vartheta},\eta)v_1+\zeta F_{\varsigma}^{\vartheta},\eta)v_2)d\zeta+\int_0^1\zeta^{\frac{\alpha}{k}-1}g(\zeta F_{\varsigma}^{\vartheta},\eta)v_1+(1-\zeta F_{\varsigma}^{\vartheta},\eta)v_2)d\zeta$$
$$\leq\int_0^1\zeta^{\frac{\alpha}{k}-1}(h(\zeta)+(1-\zeta))(g(v_1)+g(v_2))d\zeta \tag{2.3}$$

Consider,

$$\int_0^1\zeta^{\frac{\alpha}{k}-1}g\left((1-\zeta F_{\varsigma}^{\vartheta},\eta)v_1+\zeta F_{\varsigma}^{\vartheta},\eta)v_2\right)d\zeta$$

Let

$$x=(1-\zeta F_{\varsigma}^{\vartheta},\eta)v_1+\zeta F_{\varsigma}^{\vartheta},\eta)v_2$$

then

$$\int_0^1\zeta^{\frac{\alpha}{k}-1}g\left((1-\zeta F_{\varsigma}^{\vartheta},\eta)v_1+\zeta F_{\varsigma}^{\vartheta},\eta)v_2\right)d\zeta$$
$$=\frac{1}{[(v_2-v_1)F_{\varsigma}^{\vartheta},\eta]^{\frac{\alpha}{k}}}\int_{v_1}^{v_1+(v_2-v_1)F_{\varsigma}^{\vartheta},\eta}(x-v_1)^{\frac{\alpha}{k}-1}g(x)dx \tag{2.4}$$

Similarly

$$\int_0^1\zeta^{\frac{\alpha}{k}-1}g\left(\zeta F_{\varsigma}^{\vartheta},\eta)v_1+(1-\zeta F_{\varsigma}^{\vartheta},\eta)v_2\right)d\zeta$$
$$=\frac{1}{[(v_2-v_1)F_{\varsigma}^{\vartheta},\eta]^{\frac{\alpha}{k}}}\int_{v_2-(v_2-v_1)F_{\varsigma}^{\vartheta},\eta}^{v_2}(v_2-x)^{\frac{\alpha}{k}-1}g(x)dx \tag{2.5}$$

Using (2.6) and (2.7) in (2.3) we get

$$\frac{1}{[(v_2-v_1)F_{\varsigma}^{\vartheta},\eta]^{\frac{\alpha}{k}}}[\int_{v_1}^{v_1+(v_2-v_1)F_{\varsigma}^{\vartheta},\eta}(x-v_1)^{\frac{\alpha}{k}-1}g(x)dx$$
$$+\int_{v_2-(v_2-v_1)F_{\varsigma}^{\vartheta},\eta}^{v_2}(v_2-x)^{\frac{\alpha}{k}-1}g(x)dx]$$
$$\leq\int_0^1\zeta^{\frac{\alpha}{k}-1}(h(\zeta)+(1-\zeta))(g(v_1)+g(v_2))d\zeta$$

Using Definition 1.4 we get

$$\frac{k\Gamma_k(\alpha)}{2((v_2-v_1)F_{\varsigma}^{\vartheta})^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{(v_1+(v_2-v_1)F_{\varsigma}^{\vartheta})^-}g(v_1)+{}_kJ^{\alpha}_{(v_2-(v_2-v_1)F_{\varsigma}^{\vartheta})^+}g(v_2)\right]$$
$$\leq\frac{g(v_1)+g(v_2)}{2}\int_0^1\zeta^{\frac{\alpha}{k}-1}(h(\zeta)+h(1-\zeta))d\zeta$$

Which is the required result. □

**Remark 2.2.** *If we take* $h(\zeta)=\zeta, F_{\varsigma}^{\vartheta},\eta(\upsilon_2-\upsilon_1)=\upsilon_2-\upsilon_1$ *in Theorem 2.1, we will get Theorem* 3 *in [7] for* $\mu=\eta=0$.

**Remark 2.3.** *If we take* $h(\zeta)=1$, *then Theorem 2.1 yields,*

$$\frac{\Gamma_k(\alpha+k)}{4(v_2-v_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{v_2^-}g(v_1)+J^{\alpha}_{v_1^+}g(v_2)\right]\leq\frac{g(v_1)+g(v_2)}{2}$$

*which is the special case of Theorem* 3 *in [7] for* $\mu=\eta=0$.

**Theorem 2.4.** *Let* $\alpha,\rho>0$ *with a bounded sequence of real numbers* $\vartheta=(\vartheta(0),.....,\vartheta(K),.....)$ *and let* $g:[v_1,v_2]\to\mathbb{R}$ *be a* $(\psi,h)$*-Convex function then*

$$\frac{\rho^{\alpha-1}\Gamma(\alpha)}{2(v_2^{\rho}-v_1^{\rho})^{\alpha}}\left[{}^{\rho}I^{\alpha}_{([v_1^{\rho}+(F_{\varsigma}^{\vartheta},\eta)^{\rho}(v_2^{\rho}-v_1^{\rho})]^{\frac{1}{\rho}})^-}g(v_1^{\rho})+{}^{\rho}I^{\alpha}_{([v_2^{\rho}-(F_{\varsigma}^{\vartheta},\eta)^{\rho}(v_2^{\rho}-v_1^{\rho})]^{\frac{1}{\rho}})^+}g(v_2^{\rho})\right]$$
$$\leq\frac{g(v_1^{\rho})+g(v_2^{\rho})}{2}\int_0^1\zeta^{\alpha\rho-1}(h(\zeta^{\rho})+h(1-\zeta^{\rho}))d\zeta,\varsigma,\eta>0$$

*Where I is katugampola fractional integral operator and F is Raina's function.*

*Proof.*

$$g((1-(\zeta F_{\varsigma}^{\vartheta},\eta)^{\rho}v_1^{\rho})+(\zeta F_{\varsigma}^{\vartheta},\eta)^{\rho}v_2^{\rho}\rho)\leq h(1-\zeta^{\rho})g(v_1^{\rho})+h(\zeta^{\rho})g(v_2^{\rho})\tag{2.6}$$

$$g(\zeta F_{\varsigma}^{\vartheta},\eta)^{\rho}v_1^{\rho}+(1-(\zeta F_{\varsigma}^{\vartheta},\eta)^{\rho})v_2^{\rho})\leq h(\zeta^{\rho})g(v_1^{\rho})+h(1-\zeta^{\rho})g(v_2^{\rho})\tag{2.7}$$

Adding (2.6) and (2.7) ,multiply both sides by $\zeta^{\alpha\rho-1}$ and integrate from 0 to 1 we get ,

$$\int_0^1\zeta^{\alpha\rho-1}\left[g(\zeta F_{\varsigma}^{\vartheta},\eta)^{\rho}v_1^{\rho}+(1-(\zeta F_{\varsigma}^{\vartheta},\eta)^{\rho})v_2^{\rho})\right]d\zeta+$$
$$\int_0^1\zeta^{\alpha\rho-1}\left[g((1-(\zeta F_{\varsigma}^{\vartheta},\eta)^{\rho}v_1^{\rho})+(\zeta F_{\varsigma}^{\vartheta},\eta)^{\rho}v_2^{\rho})\right]d\zeta$$
$$\leq\int_0^1\zeta^{\alpha\rho-1}\left[h(1-\zeta^{\rho})+h(\zeta^{\rho})\right]\left[g(v_1^{\rho})+g(v_2^{\rho})\right]d\zeta\tag{2.8}$$

consider,

$$\int_0^1\zeta^{\alpha\rho-1}[g(\zeta F_{\varsigma}^{\vartheta},\eta)^{\rho}v_1^{\rho}+(1-(\zeta F_{\varsigma}^{\vartheta},\eta)^{\rho})v_2^{\rho})]d\zeta$$

let

$$x^\rho = (\zeta F_{\varsigma,\eta}^{\vartheta})^\rho v_1^\rho + (1-(\zeta F_{\varsigma,\eta}^{\vartheta})^\rho) v_2^\rho)$$

using the technique of integration we get ,

$$\int_0^1 \zeta^{\alpha\rho-1}[g(\zeta F_{\varsigma,\eta}^{\vartheta})^\rho v_1^\rho + (1-(\zeta F_{\varsigma,\eta}^{\vartheta})^\rho) v_2^\rho)]d\zeta$$
$$= \frac{1}{[(F_{\varsigma,\eta}^{\vartheta})^\rho (v_2^\rho - v_1^\rho)]^\alpha} \int_{[v_2^\rho-(F_{\varsigma,\eta}^{\vartheta})^\rho (v_2^\rho-v_1^\rho)]^{\frac{1}{\rho}}}^{v_2} \frac{x^{\rho-1}}{(v_2^\rho - x^\rho)^{1-\alpha}} g(x^\rho)dx \tag{2.9}$$

Similarly,

$$\int_0^1 \zeta^{\alpha\rho-1}[g((1-(\zeta F_{\varsigma,\eta}^{\vartheta})^\rho v_1^\rho) + (\zeta F_{\varsigma,\eta}^{\vartheta})^\rho v_2^\rho)]d\zeta$$
$$= \frac{1}{[(F_{\varsigma,\eta}^{\vartheta})^\rho (v_2^\rho - v_1^\rho)]^\alpha} \int_{v_1}^{v_1^\rho+(F_{\varsigma,\eta}^{\vartheta})^\rho (v_2^\rho-v_1^\rho)]^{\frac{1}{\rho}}} \frac{x^{\rho-1}}{(x^\rho - v_1^\rho)^{1-\alpha}} g(x^\rho)dx \tag{2.10}$$

using (2.9) and (2.10) in (2.8) we get

$$\frac{1}{[(F_{\varsigma,\eta}^{\vartheta})^\rho (v_2^\rho - v_1^\rho)]^\alpha} \times$$
$$\left[\int_{[v_2^\rho-(F_{\varsigma,\eta}^{\vartheta})^\rho (v_2^\rho-v_1^\rho)]^{\frac{1}{\rho}}}^{v_2} \frac{x^{\rho-1}}{(v_2^\rho - x^\rho)^{1-\alpha}} g(x^\rho)dx + \int_{v_1}^{v_1^\rho+(F_{\varsigma,\eta}^{\vartheta})^\rho (v_2^\rho-v_1^\rho)]^{\frac{1}{\rho}}} \frac{x^{\rho-1}}{(x^\rho - v_1^\rho)^{1-\alpha}} g(x^\rho)dx\right]$$
$$\leq \int_0^1 \zeta^{\alpha\rho-1}(h(\zeta^\rho)+h(1-\zeta^\rho))(g(v_1^\rho)+g(v_2^\rho))d\zeta$$

Using the Definition 1.4 we get

$$\frac{\rho^{\alpha-1}\Gamma(\alpha)}{2(v_2^\rho - v_1^\rho)^\alpha}\left[{}^\rho I^\alpha_{([v_1^\rho+(F_{\varsigma,\eta}^{\vartheta})^\rho (v_2^\rho-v_1^\rho)]^{\frac{1}{\rho}})^-} g(v_1^\rho) + {}^\rho I^\alpha_{([v_2^\rho-(F_{\varsigma,\eta}^{\vartheta})^\rho (v_2^\rho-v_1^\rho)]^{\frac{1}{\rho}})^+} g(v_2^\rho)\right]$$
$$\leq \frac{g(v_1^\rho)+g(v_2^\rho)}{2}\int_0^1 \zeta^{\alpha\rho-1}(h(\zeta^\rho)+h(1-\zeta^\rho))d\zeta$$

which is the required result. □

**Remark 2.5.** *Using the assumption that if* $h(\zeta)=\zeta$ , $(F_{\varsigma,\eta}^{\vartheta})^p(v_2^p - v_1^p) = v_2^p - v_1^p$ *in Theorem 2.4 we will get Theorem* 6 *in [7] for* $\mu=\eta=0$.

**Remark 2.6.** *If we take* $\rho=1$ *in Theorem 2.4 then we will obtain Theorem 2.1.*
*If we take* $\alpha=\rho=1$ *in Theorem 2.4 then we will obtain*

$$\frac{1}{(v_2-v_1)}\int_{v_1}^{v_2} g(x)dx \leq g(v_1)+g(v_2)$$

**Theorem 2.7.** *Let* $\alpha,\rho>0$ *with a bounded sequence of real numbers* $\vartheta=(\vartheta(0),.....,\vartheta(K),.....)$ *and let* $g:[v_1^\rho,v_2^\rho]\to\mathbb{R}$ *be a differential mapping on* $(v_1^\rho,v_2^\rho)$ *with* $0\leq v_1\leq v_2$. *Then the following inequality holds if the fractional integrals exists:*

$$\frac{g(v_1^\rho+(v_2^\rho-v_1^\rho)(F_{\varsigma,\eta}^{\vartheta})^\rho)+g(v_2^\rho-(v_2^\rho-v_1^\rho)(F_{\varsigma,\eta}^{\vartheta})^\rho)}{\alpha\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha)}{[(v_2^\rho-v_1^\rho)(F_{\varsigma,\eta}^{\vartheta})^\rho]^\alpha}\times$$
$$\left[{}^\rho I^\alpha_{([v_2^\rho-(F_{\varsigma,\eta}^{\vartheta})^\rho (v_2^\rho-v_1^\rho)]^{\frac{1}{\rho}})^+} g(v_2^\rho) + {}^\rho I^\alpha_{([v_1^\rho+(F_{\varsigma,\eta}^{\vartheta})^\rho (v_2^\rho-v_1^\rho)]^{\frac{1}{\rho}})^-} g(v_1^\rho)\right]$$
$$= \frac{[(v_2^\rho-v_1^\rho)(F_{\varsigma,\eta}^{\vartheta})^\rho]}{\alpha}\int_0^1 \zeta^{\rho(\alpha+1)-1}[g'((1-(\zeta F_{\varsigma,\eta}^{\vartheta})^\rho)v_1^\rho + (\zeta F_{\varsigma,\eta}^{\vartheta})^\rho v_2^\rho) -$$
$$g'((\zeta F_{\varsigma,\eta}^{\vartheta})^\rho v_1^\rho + (1-(\zeta F_{\varsigma,\eta}^{\vartheta})^\rho)v_2^\rho)]d\zeta;\varsigma,\eta>0$$

*Where I is katugampola fractional integral operator and F is Raina's function.*

*Proof.* consider,

$$\begin{aligned}
&\int_0^1 \zeta^{\rho(\alpha+1)-1} g'((1-(\zeta F_{\zeta}^{\vartheta},\eta)^{\rho})v_1^{\rho}+(\zeta F_{\zeta}^{\vartheta},\eta)^{\rho}v_2^{\rho})d\zeta\\
&=\int_0^1 \zeta^{\rho-1}\zeta^{\alpha\rho} g'((1-(\zeta F_{\zeta}^{\vartheta},\eta)^{\rho})v_1^{\rho}+(\zeta F_{\zeta}^{\vartheta},\eta)^{\rho}v_2^{\rho})d\zeta\\
&=\frac{1}{\rho(v_2^{\rho}-v_1^{\rho})(F_{\zeta}^{\vartheta},\eta)^{\rho}}[\zeta^{\alpha\rho} g((1-(\zeta F_{\zeta}^{\vartheta},\eta)^{\rho})v_1^{\rho}+(\zeta F_{\zeta}^{\vartheta},\eta)^{\rho}v_2^{\rho})]\mid_0^1\\
&-\frac{\alpha}{(v_2^{\rho}-v_1^{\rho})(F_{\zeta}^{\vartheta},\eta)^{\rho}}\int_0^1 \zeta^{\alpha\rho-1} g((1-(\zeta F_{\zeta}^{\vartheta},\eta)^{\rho})v_1^{\rho}+(\zeta F_{\zeta}^{\vartheta},\eta)^{\rho}v_2^{\rho})d\zeta\\
&=\frac{1}{\rho(v_2^{\rho}-v_1^{\rho})(F_{\zeta}^{\vartheta},\eta)^{\rho}}[g((1-(F_{\zeta}^{\vartheta},\eta)^{\rho})v_1^{\rho}+(F_{\zeta}^{\vartheta},\eta)^{\rho}v_2^{\rho})]\\
&-\frac{\alpha}{(v_2^{\rho}-v_1^{\rho})(F_{\zeta}^{\vartheta},\eta)^{\rho}}\int_0^1 \zeta^{\alpha\rho-1} g((1-(\zeta F_{\zeta}^{\vartheta},\eta)^{\rho})v_1^{\rho}+(\zeta F_{\zeta}^{\vartheta},\eta)^{\rho}v_2^{\rho})d\zeta \qquad (2.11)
\end{aligned}$$

As

$$\begin{aligned}
&\int_0^1 \zeta^{\alpha\rho-1}[g((1-(\zeta F_{\zeta}^{\vartheta},\eta)^{\rho}v_1^{\rho})+(\zeta F_{\zeta}^{\vartheta},\eta)^{\rho}v_2^{\rho})]d\zeta\\
&=\frac{1}{[(F_{\zeta}^{\vartheta},\eta)^{\rho}(v_2^{\rho}-v_1^{\rho})]^{\alpha}}\int_{v_1}^{v_1^{\rho}+(F_{\zeta}^{\vartheta},\eta)^{\rho}(v_2^{\rho}-v_1^{\rho})]^{\frac{1}{\rho}}}\frac{x^{\rho-1}}{(x^{\rho}-v_1^{\rho})^{1-\alpha}}g(x^{\rho})dx
\end{aligned}$$

Using above relation (2.11) becomes

$$\begin{aligned}
&\int_0^1 \zeta^{\rho(\alpha+1)-1} g'((1-(\zeta F_{\zeta}^{\vartheta},\eta)^{\rho})v_1^{\rho}+(\zeta F_{\zeta}^{\vartheta},\eta)^{\rho}v_2^{\rho})d\zeta\\
&=\frac{1}{\rho(v_2^{\rho}-v_1^{\rho})(F_{\zeta}^{\vartheta},\eta)^{\rho}}[g((1-(F_{\zeta}^{\vartheta},\eta)^{\rho})v_1^{\rho}+(F_{\zeta}^{\vartheta},\eta)^{\rho}v_2^{\rho})] \qquad (2.12)\\
&-\frac{\alpha}{[(v_2^{\rho}-v_1^{\rho})(F_{\zeta}^{\vartheta},\eta)^{\rho}]^{\alpha+1}}\int_{v_1}^{v_1^{\rho}+(F_{\zeta}^{\vartheta},\eta)^{\rho}(v_2^{\rho}-v_1^{\rho})]^{\frac{1}{\rho}}}\frac{x^{\rho-1}}{(x^{\rho}-v_1^{\rho})^{1-\alpha}}g(x^{\rho})dx \qquad (2.13)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^1 \zeta^{\rho(\alpha+1)-1} g'((\zeta F_{\zeta}^{\vartheta},\eta)^{\rho}v_1^{\rho}+(1-(\zeta F_{\zeta}^{\vartheta},\eta)^{\rho})v_2^{\rho})]d\zeta\\
&=-\frac{1}{\rho(v_2^{\rho}-v_1^{\rho})(F_{\zeta}^{\vartheta},\eta)^{\rho}}g((F_{\zeta}^{\vartheta},\eta)^{\rho}v_1^{\rho}+(1-(F_{\zeta}^{\vartheta},\eta)^{\rho})v_2^{\rho})\\
&+\frac{\alpha}{(v_2^{\rho}-v_1^{\rho})(F_{\zeta}^{\vartheta},\eta)^{\rho}}\int_0^1 \zeta^{\alpha\rho-1} g((\zeta F_{\zeta}^{\vartheta},\eta)^{\rho}v_1^{\rho}+(1-(\zeta F_{\zeta}^{\vartheta},\eta)^{\rho})v_2^{\rho})d\zeta \qquad (2.14)
\end{aligned}$$

As

$$\begin{aligned}
&\int_0^1 \zeta^{\alpha\rho-1}[g(\zeta F_{\zeta}^{\vartheta},\eta)^{\rho}v_1^{\rho}+(1-(\zeta F_{\zeta}^{\vartheta},\eta)^{\rho})v_2^{\rho})]d\zeta\\
&=\frac{1}{[(F_{\zeta}^{\vartheta},\eta)^{\rho}(v_2^{\rho}-v_1^{\rho})]^{\alpha}}\int_{[v_2^{\rho}-(F_{\zeta}^{\vartheta},\eta)^{\rho}(v_2^{\rho}-v_1^{\rho})]^{\frac{1}{\rho}}}^{v_2}\frac{x^{\rho-1}}{(v_2^{\rho}-x^{\rho})^{1-\alpha}}g(x^{\rho})dx
\end{aligned}$$

So

$$\begin{aligned}
&\int_0^1 \zeta^{\rho(\alpha+1)-1} g'((\zeta F_{\zeta,\eta}^{\vartheta})^{\rho} v_1^{\rho} + (1-(\zeta F_{\zeta,\eta}^{\vartheta})^{\rho}) v_2^{\rho})] d\zeta \\
&= -\frac{1}{\rho(v_2^{\rho}-v_1^{\rho})(F_{\zeta,\eta}^{\vartheta})^{\rho}} g((F_{\zeta,\eta}^{\vartheta})^{\rho} v_1^{\rho} + (1-(F_{\zeta,\eta}^{\vartheta})^{\rho}) v_2^{\rho}) \\
&+ \frac{\alpha}{[(v_2^{\rho}-v_1^{\rho})(F_{\zeta,\eta}^{\vartheta})^{\rho}]^{\alpha+1}} \int_{[v_2^{\rho}-(F_{\zeta,\eta}^{\vartheta})^{\rho}(v_2^{\rho}-v_1^{\rho})]^{\frac{1}{\rho}}}^{v_2} \frac{x^{\rho-1}}{(v_2^{\rho}-x^{\rho})^{1-\alpha}} g(x^{\rho}) dx
\end{aligned} \tag{2.15}$$

Subtracting (2.15) from (2.14) we get,

$$\begin{aligned}
&\int_0^1 \zeta^{\rho(\alpha+1)-1} [g'((1-(\zeta F_{\zeta,\eta}^{\vartheta})^{\rho}) v_1^{\rho} + (\zeta F_{\zeta,\eta}^{\vartheta})^{\rho} v_2^{\rho}) - \\
&g'((\zeta F_{\zeta,\eta}^{\vartheta})^{\rho} v_1^{\rho} + (1-(\zeta F_{\zeta,\eta}^{\vartheta})^{\rho}) v_2^{\rho})] d\zeta \\
&= \frac{1}{\rho(v_2^{\rho}-v_1^{\rho})(F_{\zeta,\eta}^{\vartheta})^{\rho}} [g(v_1^{\rho} + (v_2^{\rho}-v_1^{\rho})(F_{\zeta,\eta}^{\vartheta})^{\rho}) + g(v_2^{\rho} - (v_2^{\rho}-v_1^{\rho})(F_{\zeta,\eta}^{\vartheta})^{\rho}))] \\
&- \frac{\alpha}{[(v_2^{\rho}-v_1^{\rho})(F_{\zeta,\eta}^{\vartheta})^{\rho}]^{(\alpha+1)}} [\int_{[v_2^{p}-(F_{\zeta,\eta}^{\vartheta})^{p}(v_2^{p}-v_1^{p})]^{\frac{1}{p}}}^{v_2} \frac{x^{p-1}}{(v_2^{p}-x^{p})^{1-\alpha}} g(x^{\rho}) dx \\
&+ \int_{v_1}^{[v_1^{p}+(F_{\zeta,\eta}^{\vartheta})^{p}(v_2^{p}-v_1^{p})]^{\frac{1}{p}}} \frac{x^{p-1}}{(x^{p}-v_1^{p})^{1-\alpha}} g(x^{\rho}) dx]
\end{aligned}$$

Using Defination 3 we get

$$\begin{aligned}
&\int_0^1 \zeta^{\rho(\alpha+1)-1} [g'((1-(\zeta F_{\zeta,\eta}^{\vartheta})^{\rho}) v_1^{\rho} + (\zeta F_{\zeta,\eta}^{\vartheta})^{\rho} v_2^{\rho}) - \\
&g'((\zeta F_{\zeta,\eta}^{\vartheta})^{\rho} v_1^{\rho} + (1-(\zeta F_{\zeta,\eta}^{\vartheta})^{\rho}) v_2^{\rho})] d\zeta \\
&= \frac{1}{\rho(v_2^{\rho}-v_1^{\rho})(F_{\zeta,\eta}^{\vartheta})^{\rho}} [g(v_1^{\rho} + (v_2^{\rho}-v_1^{\rho})(F_{\zeta,\eta}^{\vartheta})^{\rho}) + g(v_2^{\rho} - (v_2^{\rho}-v_1^{\rho})(F_{\zeta,\eta}^{\vartheta})^{\rho}))] \\
&- \frac{\alpha\Gamma(\alpha)\rho^{\alpha-1}}{[(v_2^{\rho}-v_1^{\rho})(F_{\zeta,\eta}^{\vartheta})^{\rho}]^{(\alpha+1)}} \left[ {}^{\rho}I^{\alpha}_{([v_2^{\rho}-(F_{\zeta,\eta}^{\vartheta})^{\rho}(v_2^{\rho}-v_1^{\rho})]^{\frac{1}{\rho}})^{+}} g(v_2^{\rho}) + {}^{\rho}I^{\alpha}_{([v_1^{\rho}+(F_{\zeta,\eta}^{\vartheta})^{\rho}(v_2^{\rho}-v_1^{\rho})]^{\frac{1}{\rho}})^{-}} g(v_1^{\rho}) \right]
\end{aligned}$$

$$\begin{aligned}
&\frac{g(v_1^{\rho} + (v_2^{\rho}-v_1^{\rho})(F_{\zeta,\eta}^{\vartheta})^{\rho}) + g(v_2^{\rho} - (v_2^{\rho}-v_1^{\rho})(F_{\zeta,\eta}^{\vartheta})^{\rho})}{\alpha\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha)}{[(v_2^{\rho}-v_1^{\rho})(F_{\zeta,\eta}^{\vartheta})^{\rho}]^{\alpha}} \times \\
&\left[ {}^{\rho}I^{\alpha}_{([v_2^{\rho}-(F_{\zeta,\eta}^{\vartheta})^{\rho}(v_2^{\rho}-v_1^{\rho})]^{\frac{1}{\rho}})^{+}} g(v_2^{\rho}) + {}^{\rho}I^{\alpha}_{([v_1^{\rho}+(F_{\zeta,\eta}^{\vartheta})^{\rho}(v_2^{\rho}-v_1^{\rho})]^{\frac{1}{\rho}})^{-}} g(v_1^{\rho}) \right] \\
&= \frac{[(v_2^{\rho}-v_1^{\rho})(F_{\zeta,\eta}^{\vartheta})^{\rho}]}{\alpha} \int_0^1 \zeta^{\rho(\alpha+1)-1} [g'((1-(\zeta F_{\zeta,\eta}^{\vartheta})^{\rho}) v_1^{\rho} + (\zeta F_{\zeta,\eta}^{\vartheta})^{\rho} v_2^{\rho}) - \\
&g'((\zeta F_{\zeta,\eta}^{\vartheta})^{\rho} v_1^{\rho} + (1-(\zeta F_{\zeta,\eta}^{\vartheta})^{\rho}) v_2^{\rho})] d\zeta
\end{aligned}$$

which completes the proof. □

**Remark 2.8.** *Using the assumption* $(v_2^{\rho}-v_1^{\rho})(F_{\zeta,\eta}^{\vartheta})^{\rho} = (v_2^{\rho}-v_1^{\rho})$ *in Theorem 2.7, we will get Lemma* 3 *in [7].*

**Lemma 2.9.** *Let $\alpha > 0$ and $\rho > 0$. Let $g : [v_1^\rho, v_2^\rho] \to \mathbb{R}$ be a differential mapping on $(v_1^\rho, v_2^\rho)$ with $0 \le v_1 \le v_2$. then the following inequality holds if the fractional integrals exists:*

$$\begin{aligned}&| \frac{g(v_1^\rho)+g(v_2^\rho)}{\alpha\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha)}{2(v_2^\rho - v_1^\rho)^\alpha}[I^\alpha_{v_1^+} g(v_2^\rho) + I^\alpha_{v_2^-} g(v_1^\rho)] | \\ &\le \frac{v_2^\rho - v_1^\rho}{\alpha(\alpha+1)\rho}[| g\prime(v_1^\rho) | + | g\prime(v_2^\rho) |]\end{aligned}$$

*Proof.* The prove follows directly by using the the definition of $\psi$-Convex function and then integrating. □

**Remark 2.10.** *Lemma 2.9 is the generalization of Theorem*2.3. *in [1].*

**Theorem 2.11.** *Let $\alpha, \rho > 0$ with a bounded sequence of real numbers $\vartheta = (\vartheta(0), ....., \vartheta(K), .....)$. Let $g : [v_1^\rho, v_2^\rho] \to \mathbb{R}$ be a differential function on $(v_1^\rho, v_2^\rho)$ with $0 \le v_1 \le v_2$. If $|g\prime|^q$ be a $\psi$-Convex function for $q > 1$ then the following inequality holds:*

$$\begin{aligned}&| \frac{g(v_1^\rho + (v_2^\rho - v_1^\rho)(F^\vartheta_{\varsigma,\eta})^\rho) + g(v_2^\rho - (v_2^\rho - v_1^\rho)(F^\vartheta_{\varsigma,\eta})^\rho)}{\alpha\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha)}{[(v_2^\rho - v_1^\rho)(F^\vartheta_{\varsigma,\eta})^\rho]^\alpha} \times \\ &\left[ {}^\rho I^\alpha_{([v_2^\rho - (F^\vartheta_{\varsigma,\eta})^\rho (v_2^\rho - v_1^\rho)]^{\frac{1}{\rho}})^+} g(v_2^\rho) + {}^\rho I^\alpha_{([v_1^\rho + (F^\vartheta_{\varsigma,\eta})^\rho (v_2^\rho - v_1^\rho)]^{\frac{1}{\rho}})^-} g(v_1^\rho) \right] | \\ &\le \frac{(v_2^\rho - v_1^\rho)(F^\vartheta_{\varsigma,\eta})^\rho}{\alpha} \left(\frac{1}{\rho(\alpha+1)}\right)^{1-\frac{1}{q}} \left[\int_0^1 (\zeta^{\rho(\alpha+1)-1}[h(1-\zeta^p) \mid g'(v_1^\rho) \mid^q + h(\zeta^\rho) \mid g'(v_2^\rho) \mid^q] d\zeta\right]^{\frac{1}{q}} \\ &+ \left[\int_0^1 (\zeta^{\rho(\alpha+1)-1}[h(\zeta^\rho) \mid g'(v_1^\rho) \mid^q + h(1-\zeta^\rho) \mid g'(v_2^\rho) \mid^q] d\zeta\right]^{\frac{1}{q}}, \varsigma, \eta > 0\end{aligned}$$

*Where I is katugampola fractional integral operator and F is Raina's function.*

*Proof.* As Theorem 2.7

$$\begin{aligned}&\frac{g(v_1^\rho + (v_2^\rho - v_1^\rho)(F^\vartheta_{\varsigma,\eta})^\rho) + g(v_2^\rho - (v_2^\rho - v_1^\rho)(F^\vartheta_{\varsigma,\eta})^\rho)}{\alpha\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha)}{[(v_2^\rho - v_1^\rho)(F^\vartheta_{\varsigma,\eta})^\rho]^\alpha} \times \\ &\left[ {}^\rho I^\alpha_{([v_2^\rho - (F^\vartheta_{\varsigma,\eta})^\rho (v_2^\rho - v_1^\rho)]^{\frac{1}{\rho}})^+} g(v_2^\rho) + {}^\rho I^\alpha_{([v_1^\rho + (F^\vartheta_{\varsigma,\eta})^\rho (v_2^\rho - v_1^\rho)]^{\frac{1}{\rho}})^-} g(v_1^\rho) \right] \\ &= \frac{[(v_2^\rho - v_1^\rho)(F^\vartheta_{\varsigma,\eta})^\rho]}{\alpha} \int_0^1 \zeta^{\rho(\alpha+1)-1}[g'((1-(\zeta F^\vartheta_{\varsigma,\eta})^\rho)v_1^\rho + (\zeta F^\vartheta_{\varsigma,\eta})^\rho v_2^\rho) - \\ &g'((\zeta F^\vartheta_{\varsigma,\eta})^\rho v_1^\rho + (1-(\zeta F^\vartheta_{\varsigma,\eta})^\rho)v_2^\rho)] d\zeta\end{aligned}$$

Using Holder's inequality and the $\psi$-Convexity of $|g\prime|^q$, we get

$$\begin{aligned}&| \frac{g(v_1^\rho + (v_2^\rho - v_1^\rho)(F^\vartheta_{\varsigma,\eta})^\rho) + g(v_2^\rho - (v_2^\rho - v_1^\rho)(F^\vartheta_{\varsigma,\eta})^\rho)}{\alpha\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha)}{[(v_2^\rho - v_1^\rho)(F^\vartheta_{\varsigma,\eta})^\rho]^\alpha} \times \\ &\left[ {}^\rho I^\alpha_{([v_2^\rho - (F^\vartheta_{\varsigma,\eta})^\rho (v_2^\rho - v_1^\rho)]^{\frac{1}{\rho}})^+} g(v_2^\rho) + {}^\rho I^\alpha_{([v_1^\rho + (F^\vartheta_{\varsigma,\eta})^\rho (v_2^\rho - v_1^\rho)]^{\frac{1}{\rho}})^-} g(v_1^\rho) \right] | \\ &\le \frac{[(v_2^\rho - v_1^\rho)(F^\vartheta_{\varsigma,\eta})^\rho]}{\alpha} \int_0^1 \zeta^{\rho(\alpha+1)-1}[| g'((1-(\zeta F^\vartheta_{\varsigma,\eta})^\rho)v_1^\rho + (\zeta F^\vartheta_{\varsigma,\eta})^\rho v_2^\rho) | \\ &+ | g'((\zeta F^\vartheta_{\varsigma,\eta})^\rho v_1^\rho + (1-(\zeta F^\vartheta_{\varsigma,\eta})^\rho)v_2^\rho) |] d\zeta\end{aligned}$$

$$
\begin{aligned}
&|\frac{g(v_1^\rho+(v_2^\rho-v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho)+g(v_2^\rho-(v_2^\rho-v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho)}{\alpha\rho}-\frac{\rho^{\alpha-1}\Gamma(\alpha)}{[(v_2^\rho-v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho]^\alpha}\times\\
&\left[{}^{\rho}I^{\alpha}_{([v_2^\rho-(F_{\varsigma}^{\vartheta},\eta)^\rho(v_2^\rho-v_1^\rho)]^{\frac{1}{\bar{p}}})^+}g(v_2^\rho)+{}^{\rho}I^{\alpha}_{([v_1^\rho+(F_{\varsigma}^{\vartheta},\eta)^\rho(v_2^\rho-v_1^\rho)]^{\frac{1}{\bar{p}}})^-}g(v_1^\rho)\right]|\\
&\leq\frac{[(v_2^\rho-v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho]}{\alpha}\\
&\times[\int_0^1(\zeta^{\rho(\alpha+1)-1}\mid g'((1-(\zeta F_{\varsigma}^{\vartheta},\eta)^\rho)v_1^\rho+(\zeta F_{\varsigma}^{\vartheta},\eta)^\rho v_2^\rho)\mid^q d\zeta\\
&+\int_0^1(\zeta^{\rho(\alpha+1)-1}\mid g'((\zeta F_{\varsigma}^{\vartheta},\eta)^\rho v_1^\rho+(1-(\zeta F_{\varsigma}^{\vartheta},\eta)^\rho)v_2^\rho)\mid^q d\zeta]^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&|\frac{g(v_1^\rho+(v_2^\rho-v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho)+g(v_2^\rho-(v_2^\rho-v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho)}{\alpha\rho}-\frac{\rho^{\alpha-1}\Gamma(\alpha)}{[(v_2^\rho-v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho]^\alpha}\times\\
&\left[{}^{\rho}I^{\alpha}_{([v_2^\rho-(F_{\varsigma}^{\vartheta},\eta)^\rho(v_2^\rho-v_1^\rho)]^{\frac{1}{\bar{p}}})^+}g(v_2^\rho)+{}^{\rho}I^{\alpha}_{([v_1^\rho+(F_{\varsigma}^{\vartheta},\eta)^\rho(v_2^\rho-v_1^\rho)]^{\frac{1}{\bar{p}}})^-}g(v_1^\rho)\right]|\\
&\leq\frac{[(v_2^\rho-v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho]}{\alpha}(\int_0^1\zeta^{\rho(\alpha+1)-1}d\zeta)^{1-\frac{1}{q}}\\
&[\int_0^1(\zeta^{\rho(\alpha+1)-1}[h(1-\zeta^p)\mid g'(v_1^\rho)\mid^q+h(\zeta^\rho)\mid g'(v_2^\rho)\mid^q]d\zeta]^{\frac{1}{q}}\\
&+[\int_0^1(\zeta^{\rho(\alpha+1)-1}[h(\zeta^\rho)\mid g'(v_1^\rho)\mid^q+h(1-\zeta^\rho)\mid g'(v_2^\rho)\mid^q]d\zeta]^{\frac{1}{q}}\\
=&\frac{(v_2^\rho-v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho}{\alpha}(\frac{1}{\rho(\alpha+1)})^{1-\frac{1}{q}}[\int_0^1(\zeta^{\rho(\alpha+1)-1}[h(1-\zeta^p)\mid g'(v_1^\rho)\mid^q+h(\zeta^\rho)\mid g'(v_2^\rho)\mid^q]d\zeta]^{\frac{1}{q}}\\
&+[\int_0^1(\zeta^{\rho(\alpha+1)-1}[h(\zeta^\rho)\mid g'(v_1^\rho)\mid^q+h(1-\zeta^\rho)\mid g'(v_2^\rho)\mid^q]d\zeta]^{\frac{1}{q}}
\end{aligned}
$$

This completes the proof. □

**Remark 2.12.** *1. In Theorem 2.11 using the assumption $h(\zeta)=\zeta$ and $(F_{\varsigma}^{\vartheta},\eta)^p(\upsilon_2^p-\upsilon_1^p)=\upsilon_2^p-\upsilon_1^p$ we will get Theorem*10 *in [7] for $\mu=0$.*
*2. If we take $h(\zeta)=\zeta$ , $(F_{\varsigma}^{\vartheta},\eta)^p(\upsilon_2^p-\upsilon_1^p)=\upsilon_2^p-\upsilon_1^p$ and $q=1$ in Theorem 2.11 ,we obtain Lemma 2.9*

**Theorem 2.13.** *Let $\alpha,\rho>0$ with a bounded sequence of real numbers $\vartheta=(\vartheta(0),.....,\vartheta(K),.....)$. Let $g:[v_1^\rho,v_2^\rho]\to\mathbb{R}$ be a differential function on $(v_1^\rho,v_2^\rho)$ with $0\leq v_1\leq v_2$. If $|g\prime|^q$ is a $\psi$-Convex function for $q>1$ then*

$$
\begin{aligned}
&|\frac{g(v_1^\rho+(v_2^\rho-v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho)+g(v_2^\rho-(v_2^\rho-v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho)}{\alpha\rho}-\frac{\rho^{\alpha-1}\Gamma(\alpha)}{[(v_2^\rho-v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho]^\alpha}\times\\
&\left[{}^{\rho}I^{\alpha}_{([v_2^\rho-(F_{\varsigma}^{\vartheta},\eta)^\rho(v_2^\rho-v_1^\rho)]^{\frac{1}{\bar{p}}})^+}g(v_2^\rho)+{}^{\rho}I^{\alpha}_{([v_1^\rho+(F_{\varsigma}^{\vartheta},\eta)^\rho(v_2^\rho-v_1^\rho)]^{\frac{1}{\bar{p}}})^-}g(v_1^\rho)\right]|\\
&\leq\frac{[(v_2^\rho-v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho]}{\alpha}[\frac{1}{s\rho(\alpha+1)-s+1}]^{\frac{1}{s}}[\int_0^1(h(1-\zeta^\rho)|g'(v_1^\rho)|^q+h(\zeta^\rho)|g'(v_2^\rho)|^q)d\zeta]^{\frac{1}{q}}\\
&+[\int_0^1(h(\zeta^\rho)|g'(v_1^\rho)|^q+h(1-\zeta^\rho)|g'(v_2^\rho)|^q)d\zeta]^{\frac{1}{q}};\varsigma,\eta>0
\end{aligned}
$$

*where $\frac{1}{s}+\frac{1}{q}=1$, $I$ is katugampola fractional integral operator and $F$ is Raina's function.*

*Proof.* Using Holder's inequality and the $\psi$-Convexity of $|g\prime|^q$ in Theorem 2.7, we get

$$
\begin{aligned}
&\Big| \frac{g(v_1^\rho + (v_2^\rho - v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho) + g(v_2^\rho - (v_2^\rho - v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho)}{\alpha\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha)}{[(v_2^\rho - v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho]^\alpha} \times \\
&\left[ {}^{\rho}I^{\alpha}_{([v_2^\rho - (F_{\varsigma}^{\vartheta},\eta)^\rho (v_2^\rho - v_1^\rho)]^{\frac{1}{\rho}})^+} g(v_2^\rho) + {}^{\rho}I^{\alpha}_{([v_1^\rho + (F_{\varsigma}^{\vartheta},\eta)^\rho (v_2^\rho - v_1^\rho)]^{\frac{1}{\rho}})^-} g(v_1^\rho) \right] \Big| \\
&\leq \frac{[(v_2^\rho - v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho]}{\alpha} \int_0^1 \zeta^{\rho(\alpha+1)-1} [\,|\, g'((1-(\zeta F_{\varsigma}^{\vartheta},\eta)^\rho)v_1^\rho + (\zeta F_{\varsigma}^{\vartheta},\eta)^\rho v_2^\rho)\,| \\
&+ |\, g'((\zeta F_{\varsigma}^{\vartheta},\eta)^\rho v_1^\rho + (1-(\zeta F_{\varsigma}^{\vartheta},\eta)^\rho)v_2^\rho)\,|] d\zeta \\
&\leq \frac{[(v_2^\rho - v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho]}{\alpha} \Big( \int_0^1 \zeta^{s\rho(\alpha+1)-s)^{\frac{1}{s}}} [ \Big( \int_0^1 |\, g'((1-(\zeta F_{\varsigma}^{\vartheta},\eta)^\rho)v_1^\rho + (\zeta F_{\varsigma}^{\vartheta},\eta)^\rho v_2^\rho)\,|^q \, d\zeta \Big)^{\frac{1}{q}} \\
&+ \Big( \int_0^1 |\, g'((\zeta F_{\varsigma}^{\vartheta},\eta)^\rho v_1^\rho + (1-(\zeta F_{\varsigma}^{\vartheta},\eta)^\rho)v_2^\rho)\,|^q \Big) d\zeta \Big)^{\frac{1}{q}}] \\
&\leq \frac{[(v_2^\rho - v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho]}{\alpha} [\frac{1}{s\rho(\alpha+1)-s+1}]^{\frac{1}{s}} [\int_0^1 (h(1-\zeta^\rho)|g'(v_1^\rho)|^q + h(\zeta^\rho)|g'(v_2^\rho)|^q) d\zeta]^{\frac{1}{q}} \\
&+ [\int_0^1 (h(\zeta^\rho)|g'(v_1^\rho)|^q + h(1-\zeta^\rho)|g'(v_2^\rho)|^q) d\zeta]^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&\Big| \frac{g(v_1^\rho + (v_2^\rho - v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho) + g(v_2^\rho - (v_2^\rho - v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho)}{\alpha\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha)}{[(v_2^\rho - v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho]^\alpha} \times \\
&\left[ {}^{\rho}I^{\alpha}_{([v_2^\rho - (F_{\varsigma}^{\vartheta},\eta)^\rho (v_2^\rho - v_1^\rho)]^{\frac{1}{\rho}})^+} g(v_2^\rho) + {}^{\rho}I^{\alpha}_{([v_1^\rho + (F_{\varsigma}^{\vartheta},\eta)^\rho (v_2^\rho - v_1^\rho)]^{\frac{1}{\rho}})^-} g(v_1^\rho) \right] \Big| \\
&\leq \frac{[(v_2^\rho - v_1^\rho)(F_{\varsigma}^{\vartheta},\eta)^\rho]}{\alpha} [\frac{1}{s\rho(\alpha+1)-s+1}]^{\frac{1}{s}} [\int_0^1 (h(1-\zeta^\rho)|g'(v_1^\rho)|^q + h(\zeta^\rho)|g'(v_2^\rho)|^q) d\zeta]^{\frac{1}{q}} \\
&+ [\int_0^1 (h(\zeta^\rho)|g'(v_1^\rho)|^q + h(1-\zeta^\rho)|g'(v_2^\rho)|^q) d\zeta]^{\frac{1}{q}}
\end{aligned}
$$

This completes the proof. □

**Remark 2.14.** *Using the assumption* $h(\zeta) = \zeta$ *and* $(F_{\varsigma}^{\vartheta},\eta)^p(\upsilon_2^p - \upsilon_1^p) = \upsilon_2^p - \upsilon_1^p$ *in Theorem 2.13 we will get Theorem*11 *in [7] for* $\mu = 0$

## 3. Application to Special Means

[13] For some positive real numbers $v_1, v_2 (v_1 \neq v_2), v_2 \geq v_1$, we shall assume the following special means.
(1) The arithmetic mean:

$$A = A(v_1, v_2) = \frac{v_1 + v_2}{2}$$

(2) The n-arithmetic mean for $n \in \mathbb{R}/(-1,0)$:

$$A_n = A_n(v_1, v_2) = \frac{v_1^n + v_2^n}{2}$$

(3) The logarithmic mean:

$$L = L(v_1.v_2) = \frac{v_2 - v_1}{\ln v_2 - \ln v_1}, v_1 \neq v_2$$

(4) The n-logarithmic mean for $n \in \mathbb{R}/(-1,0)$:

$$L_n = L_n(v_1.v_2) = \frac{v_2^{n+1} - v_1^{n+1}}{(n+1)(\ln v_2 - \ln v_1)}, v_1 \neq v_2$$

**Proposition 3.1.** *For $v_1, v_2 \in \mathbb{R}$ with $v_2 > v_1 > 0$, then we have the following inequality:*

$$L_n(v_1, v_2) + \frac{2}{(n+1)(v_2 - v_1)^2}\beta(2, n+1)A_{(n+1)}(v_1, v_2) \leq A(v_1, v_2)$$

*Proof.* In Theorem 1.9 with $h(\zeta) = \zeta$, $F_{\varsigma,\eta}^{\vartheta}(v_2 - v_1) = v_2 - v_1$ and a function $g : [v_1, v_2] \longrightarrow \mathbb{R} = x^n, (n \in N)$ we will get the required inequality. □

**Proposition 3.2.** *For $v_1, v_2 \in \mathbb{R}$ with $v_2 > v_1 > 0$, then*

$$L_n(v_1^\rho, v_2^\rho) + \frac{2\rho}{(n+1)(v_2^\rho - v_1^\rho)^2}\beta(2, n+2)A_{(n+2)}(v_1^\rho, v_2^\rho) \leq A_n(v_1^\rho, v_2^\rho)$$

*Proof.* In Theorem 2.1 with $h(\zeta) = \zeta$, $(F_{\varsigma,\eta}^{\vartheta})^\rho(v_2^\rho - v_1^\rho) = v_2^\rho - v_1^\rho$ and a function $g : [v_1, v_2] \longrightarrow \mathbb{R} = x^n, (n \in N)$ we will have the above inequality. □

## 4. Conclusion

In this article, we consider a new generalized form of convex functions which is known as $\psi$-convex functions. In the achievement of our target, we have derived some new inequalities that deduced from the definition of different fractional integral operators and the use of $\psi$-Convex functions. The results for $\psi$-convex functions generalizing and enhancing the results and inequalities which are already existed in mathematical literature. Others antecedently got for the Riemann-liouville fractional integrals. Specifically, our results focusing on the most popular integral inequality which is known as Hermite-Hadamard inequality and three famous fractional integrals. These results will serve as a motivation and prove benefical for future work in this field. For the suitable selection of the function $h(\zeta)$ one can discover so many numerous results as particular cases. This shows that the idea of generalized convexity is extremely wide and unifying. It is expected that this article will provide new directions and ideas in fractional operators, special functions and related fields.

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