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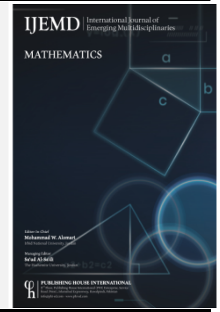

Reduce Differential Transform Method for Analytical Approximation of Fractional Delay Differential Equation

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Abstract

The study of an entirely new class of differential equations known as delay differential equations or difference differential equations has resulted from the development and application of automatic control systems (DDEs). Time delays are virtually always present in any system that uses feedback control. A time delay is required because it takes a finite amount of time to sense information and then react to it. In this investigation, we find the solution of fractional delay differential equations by using the reduced differential transform method. The results so obtained are in the form of series. It is observed that the proposed technique is accurate and convergent. MAPLE 17 is used to illustrate the results graphically.

Keywords: Pantograph equations; Delay differential equation; Fractional differential equation; Reduced differential transform method.

1. Introduction

Gorenflo and Mainardi [1] provided the fundamental definitions of functional calculus and its applications. In their lectures, they introduce the linear operators of functional integration and functional differentiation in the context of Riemann-Liouville fractional calculus. They paid special attention to the method of Laplace transforms for addressing these operators in a way that applied scientists could understand, avoiding pointless generalizations and unnecessary mathematical rigour. They used this method to provide analytical solutions to the simplest linear integral and differential equations of fractional order. They go over some of the author's applications of fractional calculus to some basic problems in continuum and statistical mechanics. The difficulties in continuum mechanics related to the mathematical modelling of

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viscoelastic bodies and the unstable motion of a particle in a viscous fluid, known as the Basset problem. In the former analysis, fractional calculus leads them to introduce intermediate models of viscoelasticity, which generalise the classical spring-dashpot models. The latter analysis leads them to introduce a hydrodynamic model suitable for revisiting the classical theory of Brownian motion, which is a relevant topic in statistical mechanics. They explained the large tails in the velocity correlation and displacement variance using fractional calculus methods. They discuss the fractional diffusion-wave equation, which derived from the classical diffusion equation by substituting a fractional derivative of order with $0 < \alpha < 2$ for the first-order time derivative [2].

R. Khalil [3] and colleagues defined fractional derivative and fractional integral in a novel way. The form of the definition demonstrates that it is the most natural and fruitful definition. The definition for $0 \leq \alpha < 1$ corresponds to the classical definitions for polynomials (up to a constant). Furthermore, if $\alpha = 1$, the definition corresponds to the classical definition of the first derivative. They discussed several applications of fractional differential equations. [3]. In recent years, fractional calculus has been recognized as a powerful tool for studying the behaviour of numerous phenomena in science and engineering. In [4-7], some examples of fractional calculus applications are provided. Because of its widespread use, fractional differential equations have garnered the most attention of all fractional models. [8-10] discuss some outstanding research on the theoretical study of this family of equations.

Following World War I, the development and use of automated control systems resulted in research into an entirely new class of differential equations known as delay differential equations or difference differential equations (DDE). Time delay is nearly always present in any system that uses feedback control. A time delay occurs when a certain amount of time is required to detect information and then react to it. Severe stability issues, on the other hand, emerge when many systems must be controlled at the same time. Pilot-induced oscillations (PIO), for example, are accidental prolonged oscillations caused by the pilot's efforts to control the aircraft [11]. Fatemah and Dehghan addressed the various applications of the Delay Differential equation in science and engineering. They occur when the rate of change of a time-dependent process in its mathematical modelling is governed not only by its current state but also by a specific historical state. Recent research in domains as diverse as biology, economics, control, and electrodynamics has revealed that DDEs play a significant role in describing a wide range of phenomena. They are particularly important when ODE-based models fail. In their study, they provided the solution of a delay differential equation using a homotopy perturbation approach, followed by various numerical demonstrations. These findings show that the suggested strategy is both successful and straightforward to implement [12].

Asymptotic methods for singularly perturbed delay differential equations are more difficult to construct than approaches for ordinary differential equations. The applications of asymptotic approaches range from the obvious to the bizarre, and they demonstrate the general technical challenges that delay equations offer for the core methodology of the applied mathematician [13]. The method can easily apply to linear or nonlinear problems and is capable of reducing the size of computational work. In this work, additionally, analytical form solutions of two diffusion problems have been obtained and the solutions are compared very well with those obtained by decomposition method. Zhou [14], who solved linear and nonlinear initial value problems in electric circuit analysis, first introduced the concept of differential transform. "The differential transform is an iterative approach for generating Taylor series solutions of differential equations," says Jang [15]. This strategy decreases the size of the computational domain and is easily adaptable to a wide range of situations. Most of the researcher worked on the concept of fractional calculus and the solution of fractional differential equations, which can be seen [16-22].

Gusu et al. [23] discussed the new reduced differential transform technique (RDTM) to calculate analytical and semianalytical approximation solutions to fractional order Airy's ordinary differential equations and fractional order Airy's and Airy's type partial differential equations according to specific beginning conditions. They found the proposed scheme is reliable, efficient to handled fractional type Airy's type differential equations. Thangavelu and Padmasekaran [24] researched fractional-order partial differential equations with proportional delay, including modified Burger equations with proportional delay. Natural transform decomposition technique solutions are achieved in series form for both fractional and integer orders, demonstrating the proposed approach's increased convergence. Tahir et al. [25] solved the inhomogeneous fractional Cauchy–Riemann equation in both space and time variables using analytic Cauchy data using the vectorial fractional reduced differential transformed approach. They discovered that the solutions correspond well with the precise answer for $\alpha = \beta = 1$. Shah et al. [26] investigated fractional-order partial differential equations with proportional delay, including modified Burger equations with proportional delay. They solved these equations using the Natural transform decomposition technique. Natural transform decomposition technique solutions are achieved in series form for both fractional and integer orders, demonstrating the proposed approach's increased convergence. Many other researcher found the numerical solution of the fractional order differential equation with time delay, which can be seen in [27-36]. Here we investigated the fractional delay differential equation by using the reduced differential transform method by considering different conditions and check the reliability and convergence of the proposed scheme.

2. Definitions

The following are some fundamental definitions of the RDT Method.

2.1 Definition [37]

If the function $u(x,t)$ is analytic and continuously differentiated with respect to t, then let

$$U_k(x) = \frac{1}{k!}\left(\frac{\partial^k}{\partial x^k}u(x,t)\right)_{t=0}, \tag{1}$$

The converted functions are the $t-$dimensional spectrum functions$U_K(x)$. The original functions are denoted by the lower case $u(x,t)$, whereas the modified functions are denoted by the upper case $U(x,t)$.

2.2 Definition [37]

$U_k(x)$ has a differential inverse transform that is defined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k, \tag{2}$$

When equations (1) and (2) are combined, the following result may be obtained:

$$u(x,t) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\partial^k}{\partial x^k}u(x,t)\right)_{t=0} t^k, \tag{3}$$

The notion of the reduced differential transform technique is derived from the power series expansion of a function, according to the preceding definitions. The reduced differential transform technique is used to conduct the following mathematical tasks.

Table 1. REDUCED DIFFERENTIAL TRANSFORM (RDT) METHOD [37].

Function from	Transformed form
$u(x,t)$	$U_k(x) = \frac{1}{k!}\left(\frac{\partial^k}{\partial x^k}u(x,t)\right)_{t=0},$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = cu(x,t)$	$W_k(x) = cU_k(x)$

$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{n=0}^{k} U_n V_{k-n} = \sum_{n=0}^{k} V_n U_{k-n}$
$w(x,t) = \frac{\partial^n}{\partial y^n} u(x,t)$	$W_k(x) = (k+1)(k+2)\dots(k+n)U_{k+n}(x)$
$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = x^m U_{k-n}(x)$
$w(x,t) = x^m t^n$	$\mathrm{W_k(x) = x^m \delta(k-n), \delta(k)} = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$
$w(x,t) = \frac{\partial^{N\alpha}}{\partial x^{N\alpha}} u(x,t)$	$W_k(x) = \frac{\Gamma(k\alpha + N\alpha + 1)}{\Gamma(k\alpha + 1)} U_{k+N}(x)$
$w(x,t) = e^{ax}$	$W_k(x) = \frac{a^k}{k!}$
$w(x,t) = \sin(x)$	$W_k(x) = -\frac{1}{k!}\left(\frac{\partial^k}{\partial x^k}\sin(x)\right)$
$w(x,t) = \cos(x)$	$W_k(x) = -\frac{1}{k!}\left(\frac{\partial^k}{\partial x^k}\cos(x)\right)$
$w(x,t) = e^{-x+a}$	$W_k(x) = -\frac{1}{k!}(-1)^k e^a$
$w(x,t) = f(x-r). r \geq 1$	$W_k(x) = \sum_{\mathrm{h_1=k}}^{\mathrm{N}} (-1)^{\mathrm{h_1-k}} \binom{\mathrm{h_1}}{\mathrm{k}} (\mathrm{r})^{\mathrm{h_1-k}} \mathrm{Y(h_1)}, \mathrm{N} \to \infty$

3. Implementation of RDTM

Eight multi-pantograph delay differential equations presented in this part to demonstrate the RDTM's efficiency by using Maple 17. The series solution obtained is compared to exact solution and found to be in good agreement with one another.

3.1 Example-A [18]

Consider the first-order LDDE of the pantograph type

$$\frac{dy(x)}{dx} = \left(\frac{1}{2}\right) e^{\frac{x}{2}} y\left(\frac{x}{2}\right) + \left(\frac{1}{2}\right) y(x), 0 < x < 1, \quad (4)$$

subject to the condition

$$y(0) = 1,$$

On equation (4), we obtain the following Recurrence relation using the fractional differential transform presented in table 1

$$Y(k+1) = \frac{\Gamma(k\alpha + 1)}{2\Gamma(k\alpha + \alpha + 1)}\left[\sum_{l=0}^{k} \frac{1}{2^k l!} Y(k-l) + Y(k)\right], k \geq 0, Y(0) = 1, \quad (5)$$

Using the recurrence relation (5), we can see that

$$Y(1) = \frac{1}{\Gamma(\alpha + 1)},$$

$$Y(2) = \frac{1}{2} \frac{\Gamma(\alpha + 1)\left(\frac{1}{2} + \frac{3}{2\Gamma(\alpha + 1)}\right)}{\Gamma(2\alpha + 1)},$$

$$Y(3) = \frac{1}{2} \frac{\Gamma(2\alpha + 1)\left(\frac{1}{8} + \frac{5}{8} \frac{\Gamma(\alpha + 1)\left(\frac{1}{2} + \frac{3}{2\Gamma(\alpha + 1)}\right)}{\Gamma(2\alpha + 1)} + \frac{1}{4\Gamma(\alpha + 1)}\right)}{\Gamma(3\alpha + 1)}, \ldots$$

By using the inverse differential transform of Y (k), which is

$$y(x) = \sum_{k=0}^{\infty} Y(k) x^{k\alpha};$$

As a result, we arrive to the following series solution:

$$y(x) = \sum_{k=0}^{\infty} Y(k) x^{k\alpha}$$

$$= 1 + \left(\frac{1}{\Gamma(\alpha+1)}\right) x^{\alpha} + \left(\frac{1}{2} \frac{\Gamma(\alpha+1)\left(\frac{1}{2} + \frac{3}{2\Gamma(\alpha+1)}\right)}{\Gamma(2\alpha+1)}\right) x^{2\alpha}$$

$$+ \left(\frac{1}{2} \frac{\Gamma(2\alpha+1)\left(\frac{1}{8} + \frac{5}{8} \frac{\Gamma(\alpha+1)\left(\frac{1}{2} + \frac{3}{2\Gamma(\alpha+1)}\right)}{\Gamma(2\alpha+1)} + \frac{1}{4\Gamma(\alpha+1)}\right)}{\Gamma(3\alpha+1)}\right) x^{3\alpha} + \cdots$$

$y(x) = e^{x}$ is exact solution.

The graphs below show for various fractional values of α

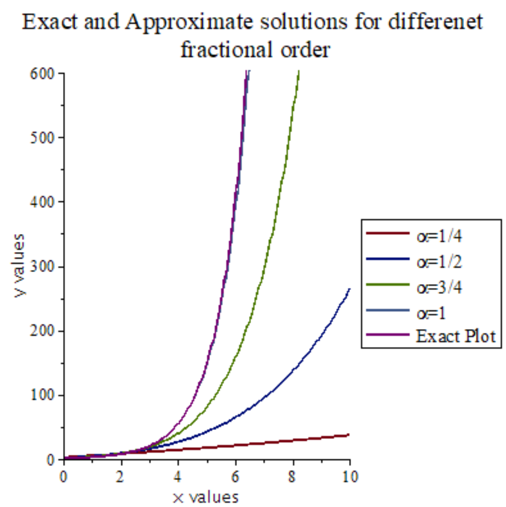

Fig. 1. For different fractional orders, a graph of exact and approximate solutions.

3.2 Example-B [19]

Consider the linear pantograph equation

$$u'(t) = -u(t) + \frac{1}{10} u\left(\frac{t}{5}\right) - \frac{1}{10} e^{-\frac{t}{5}}, 0 \le t \le 1, \tag{6}$$

depends on the condition

$$u(0) = 1.$$

On equation (6), we obtain the following Recurrence relation using the fractional differential transform presented in table 1:

$$U(k+1) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left[-U(k) + \frac{1}{10}\left(\frac{1}{5}\right)^k U(k) - \frac{1}{10}\frac{\left(\frac{1}{5}\right)^k}{k!}\right], k \geq 0, Y(0) = 1, \tag{7}$$

Utilizing the recurrence relation, we can see that

$$U(1) = -\frac{1}{\Gamma(\alpha+1)},$$

$$U(2) = \frac{\Gamma(\alpha+1)\left(\frac{49}{50\Gamma(\alpha+1)} + \frac{1}{50}\right)}{\Gamma(2\alpha+1)},$$

$$U(3) = \frac{\Gamma(2\alpha+1)\left(-\frac{249}{250}\frac{\Gamma(\alpha+1)\left(\frac{49}{50\Gamma(\alpha+1)} + \frac{1}{50}\right)}{\Gamma(2\alpha+1)} - \frac{1}{500}\right)}{\Gamma(3\alpha+1)}, \ldots$$

By using the inverse differential transform of Y (k), which is;

$$u(t) = \sum_{k=0}^{\infty} U(k)t^{k\alpha}.$$

As a result, we arrive to the following series solution

$$u(t) = \sum_{k=0}^{\infty} U(k)t^{k\alpha} = 1 - \frac{1}{\Gamma(\alpha+1)}t^{\alpha} + \frac{\Gamma(\alpha+1)\left(\frac{49}{50\Gamma(\alpha+1)} + \frac{1}{50}\right)}{\Gamma(2\alpha+1)}t^{2\alpha}$$
$$+ \frac{\Gamma(2\alpha+1)\left(-\frac{249}{250}\frac{\Gamma(\alpha+1)\left(\frac{49}{50\Gamma(\alpha+1)} + \frac{1}{50}\right)}{\Gamma(2\alpha+1)} - \frac{1}{500}\right)}{\Gamma(3\alpha+1)}t^{3\alpha}$$
$$+ \cdots$$

$u(t) = \exp(-t)$ is exact solution. The graphs below show for various fractional values of α

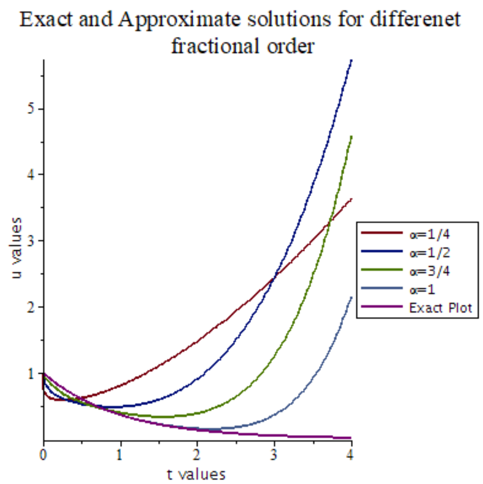

Fig. 2. For different fractional orders, a graph of exact and approximate solutions.

3.3 Example-C [20]

Consider the equation for the Linear 2nd-order Multi-Pantograph.

$$\frac{d^2y(x)}{dx^2} = \frac{3}{4}y(x) + y\left(\frac{x}{2}\right) - x^2 + 2, 0 \leq x \leq 1, \tag{8}$$

subject to the condition

$$y(0) = 0, \frac{dy(0)}{dx} = 0.$$

On equation (84.5), we obtain the following Recurrence relation using the fractional differential transform presented in table 1

$$Y(k+2) = \frac{\Gamma(k\alpha + 1)}{\Gamma(k\alpha + 2\alpha + 1)}\left(\frac{3}{4}Y(k) + \frac{1}{2^k}Y(k) - \delta(k-2) + 2\delta(k)\right), k \geq 0, Y(0) = 0, Y(1) = 0. \tag{9}$$

Utilizing the recurrence relation(9), we can see that

$$Y(2) = \frac{2}{\Gamma(2\alpha + 1)}, Y(3) = 0,$$

$$Y(4) = \left(\frac{\Gamma(2\alpha + 1)}{\Gamma(4\alpha + 1)}\right)\left(\frac{2}{\Gamma(2\alpha + 1)} - 1\right),$$

$$Y(5) = 0,$$

$$Y(6) = \frac{13}{16}\left(\frac{\Gamma(4\alpha+1)}{\Gamma(6\alpha+1)}\right)\left(\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\right)\left(\frac{2}{\Gamma(2\alpha+1)} - 1\right),$$

$$Y(7) = 0, \ldots$$

By using the inverse differential transform of Y (k), which is

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^{k\alpha}.$$

As a result, we arrive to the following series solution:

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} Y(k)x^{k\alpha} \\ &= \left(\frac{2}{\Gamma(2\alpha+1)}\right)x^{2\alpha} + \left(\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\right)\left(\frac{2}{\Gamma(2\alpha+1)} - 1\right)x^{4\alpha} \\ &+ \frac{13}{16}\left(\frac{\Gamma(4\alpha+1)}{\Gamma(6\alpha+1)}\right)\left(\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\right)\left(\frac{2}{\Gamma(2\alpha+1)} - 1\right)x^{6\alpha} + \cdots \end{aligned}$$

$y(x) = x^2$ is exact solution. The graphs below show for various fractional values of α.

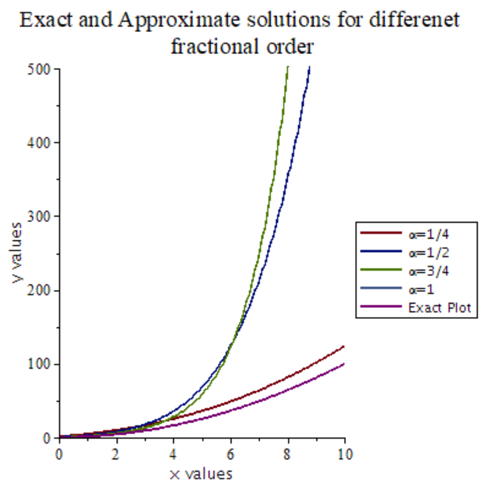

Fig. 3 For different fractional orders, a graph of exact and approximate solutions.

3.4 Example-D [21]

Consider the second order pantograph type delay differential equation.

$$\frac{d^2y(x)}{dx^2} = \frac{dy}{dx} + 2y^2\left(\frac{x}{2}\right) - 3y\left(\frac{x}{3}\right)y\left(\frac{x}{2}\right) + 2x + 2, \tag{10}$$

Subject to the conditions

$$y(0) = 0, \frac{dy(0)}{dx} = 0.$$

On equation (9), we obtain the following Recurrence relation using the fractional differential transform presented in table 1.

$$Y(k+2) = \frac{\Gamma(k\alpha + 1)}{\Gamma(k\alpha + 2\alpha + 1)}\Bigg((k+1)Y(k+1) + 2\sum_{l=0}^{k}\frac{1}{2^l}Y(l)\frac{1}{2^{k-l}}Y(k-l) - 3\sum_{l=0}^{k}\frac{1}{3^l}Y(l)\frac{1}{3^{k-l}}Y(k-l) - 2\delta(k-1) + 2\delta(k)\Bigg), k \geq 0, Y(0) = 0, Y(1) = 0, \tag{11}$$

Utilizing the recurrence relation (11), we can see that

$$Y(2) = \frac{2}{\Gamma(2\alpha + 1)}, Y(3) = \frac{\Gamma(\alpha + 1)\left(\frac{4}{\Gamma(2\alpha + 1)} - 2\right)}{\Gamma(3\alpha + 1)},$$

$$Y(4) = \frac{3\Gamma(2\alpha + 1)\Gamma(\alpha + 1)\left(\frac{4}{\Gamma(2\alpha + 1)} - 2\right)}{\Gamma(4\alpha + 1)\Gamma(3\alpha + 1)},$$

$$Y(5) = \frac{12\Gamma(2\alpha + 1)\Gamma(\alpha + 1)\left(\frac{4}{\Gamma(2\alpha + 1)} - 2\right)}{\Gamma(5\alpha + 1)\Gamma(4\alpha + 1)}, \ldots$$

By using the inverse differential transform of Y (k), which is

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^{k\alpha}.$$

As a result, we arrive to the following series solution

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^{k\alpha}$$

$$= \frac{2}{\Gamma(2\alpha+1)}x^{2\alpha} + \frac{\Gamma(\alpha+1)\left(\frac{4}{\Gamma(2\alpha+1)} - 2\right)}{\Gamma(3\alpha+1)}x^{3\alpha}$$
$$+ \frac{3\Gamma(2\alpha+1)\Gamma(\alpha+1)\left(\frac{4}{\Gamma(2\alpha+1)} - 2\right)}{\Gamma(4\alpha+1)\Gamma(3\alpha+1)}x^{4\alpha}$$
$$+ \frac{12\Gamma(2\alpha+1)\Gamma(\alpha+1)\left(\frac{4}{\Gamma(2\alpha+1)} - 2\right)}{\Gamma(5\alpha+1)\Gamma(4\alpha+1)}x^{5\alpha} + \cdots$$

$y(x) = x^2$ is exact solution. The graphs below show for various fractional values of α.

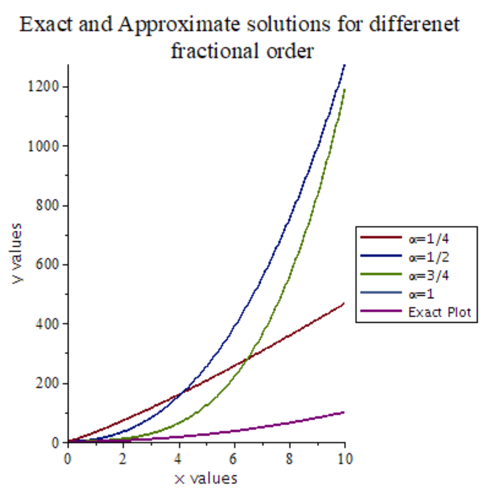

Fig. 4. For different fractional orders, a graph of exact and approximate solutions.

3.5 Example-E [18]

The linear pantograph equation is as follows

$$u''(t) = 4e^{-\frac{t}{2}}\sin\left(\frac{t}{2}\right)u\left(\frac{t}{2}\right), 0 \leq t \leq 1, \quad (12)$$

subject to the conditions

$$u(0) = 1, u'(0) = -1.$$

On equation (11), we obtain the following Recurrence relation using the fractional differential transform presented in table 1.

$$U(k+2) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+2\alpha+1)}\left[4\sum_{r_2=0}^{k}\sum_{r_1=0}^{r_2}\left(\frac{-1}{2}\right)^{r_1}\left(\frac{1}{2}\right)^{k-r_1}\frac{1}{r_1!\,(r_2-r_1)!}\sin\left(\frac{(r_2-r_1)\pi}{2}\right)U(k-r_2)\right], k \geq 0, U(0) = 1, U(1) = -1 \quad (13)$$

Utilizing the recurrence relation (12), we can see that

$$U(2) = 0, U(3) = \frac{2\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}, U(4) = -\frac{2\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}, \ldots$$

By using the inverse differential transform of Y (k), which is;;

$$u(t) = \sum_{k=0}^{\infty} U(k)t^{k\alpha};$$

As a result, we arrive to the following series solution:

$$u(t) = \sum_{k=0}^{\infty} U(k)t^{k\alpha} = 1 - t^{\alpha} + \frac{2\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha} - \frac{2\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}t^{4\alpha} + \frac{2}{3}\frac{\Gamma(3\alpha+1)}{\Gamma(5\alpha+1)}t^{5\alpha} + \cdots$$

$u(t) = \exp(-t)\cos(t)$ is exact solution.

The graphs below show for various fractional values of α.

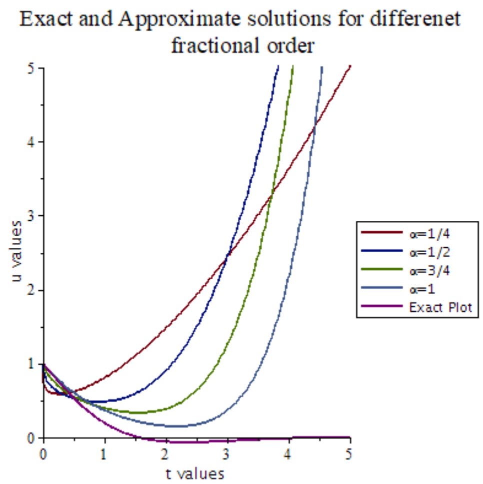

Fig. 5. For different fractional orders, a graph of exact and approximate solutions.

3.6 Example-F [21]

The nonlinear DDE with unbounded delay is considered.

$$\frac{d^2y}{dx^2} = 1 - 2y^2\left(\frac{x}{2}\right), \quad 0 \le x \le 1, \qquad (14)$$

subject to the conditions

$$y(0) = 1, \frac{dy(0)}{dx} = 0.$$

On equation (13), we obtain the following Recurrence relation using the fractional differential transform presented in table 1

$$Y(k+2) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+2\alpha+1)}\left(\delta(k) - 2\sum_{l=0}^{k}\frac{1}{2^k}Y(l)Y(k-l)\right), k \ge 0, Y(0) = 1, Y(1) = 0. \qquad (14)$$

Utilizing the recurrence relation (14), we can see that

$$Y(2) = -\frac{1}{\Gamma(2\alpha+1)}, Y(3) = 0, Y(4) = \left(\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\right)\left(\frac{1}{\Gamma(2\alpha+1)}\right), Y(5) = 0, \ldots$$

By using the inverse differential transform of Y (k), which is;;

$$y(x)=\sum_{k=0}^{\infty}Y(k)x^{k\alpha};$$

As a result, we arrive to the following series solution:

$$y(x)=\sum_{k=0}^{\infty}Y(k)x^{k\alpha}=1-\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\left(\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\right)\left(\frac{1}{\Gamma(2\alpha+1)}\right)x^{4\alpha}+\cdots$$

$y(x)=\cos(x)$ is exact solution. The graphs below show for various fractional values of α.

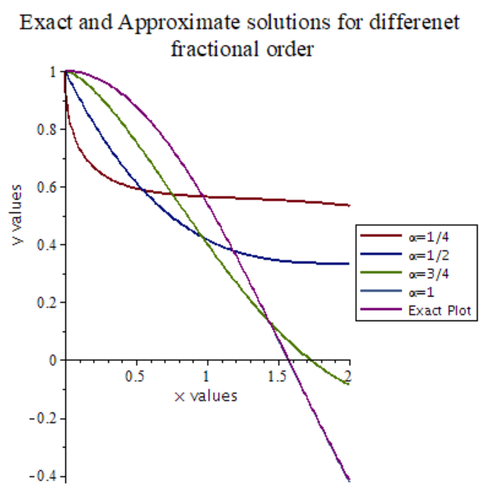

Fig. 6. For different fractional orders, a graph of exact and approximate solutions.

3.7 Example-G [21]

Consider the following third-order nonlinear Pantograph DDE

$$\frac{d^3y}{dx^3}=-1+2y^2\left(\frac{x}{2}\right), \tag{15}$$

Subject to the condition

$$y(0)=0, \frac{dy(0)}{dx}=1, \frac{d^2y(0)}{dx^2}=0,$$

On equation (15), we obtain the following Recurrence relation using the fractional differential transform presented in table 1.

$$Y(k+3)=\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+3\alpha+1)}\left(-\delta(k)+2\sum_{l=0}^{k}\frac{1}{2^{k}}Y(l)Y(k-l)\right),k\geq 0,Y(0)=0,Y(1)=1,Y(2)=0 \quad (16)$$

Utilizing the recurrence relation (16), we can see that

$$Y(3)=-\frac{1}{\Gamma(3\alpha+1)},Y(4)=0,Y(5)=\frac{1}{2}\frac{\Gamma(2\alpha+1)}{\Gamma(5\alpha+1)},Y(6)=0,Y(7)=-\frac{1}{4}\frac{\Gamma(4\alpha+1)}{\Gamma(7\alpha+1)\Gamma(3\alpha+1)},Y(8)=0,\dots$$

By using the inverse differential transform of Y (k), which is

$$y(x)=\sum_{k=0}^{\infty}Y(k)x^{k\alpha}.$$

As a result, we arrive to the following series solution:

$$y(x)=\sum_{k=0}^{\infty}Y(k)x^{k\alpha}=x^{\alpha}-\frac{1}{\Gamma(3\alpha+1)}x^{3\alpha}+\frac{1}{2}\frac{\Gamma(2\alpha+1)}{\Gamma(5\alpha+1)}x^{5\alpha}-\frac{1}{4}\frac{\Gamma(4\alpha+1)}{\Gamma(7\alpha+1)\Gamma(3\alpha+1)}x^{7\alpha}+\cdots$$

$y(x)=\sin(x)$ is exact solution. The graphs below show for various fractional values of α.

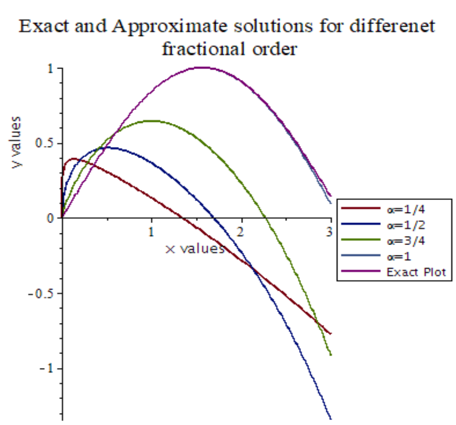

Fig. 7. For different fractional orders, a graph of exact and approximate solutions.

3.8 Example-H [18, 20]

Consider the 3rd-order Constant LDDE

$$\frac{d^3y}{dx^3} + y(x) + y(x-0.3) = e^{-x+0.3}, 0 < x < 1, \tag{17}$$

subject to the conditions

$$y(0) = 1, \frac{dy(0)}{dx} = -1, \frac{d^2y(0)}{dx^2} = 1,$$

On equation (17), we obtain the following Recurrence relation using the fractional differential transform presented in table 1.

$$\begin{aligned} Y(k+3) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+3\alpha+1)} & \left(\frac{1}{k!}(-1)^k e^{0.3} - Y(k) \right. \\ & \left. - \sum_{h_1=k}^{10} (-1)^{h_1-k} \binom{h_1}{k} (0.3)^{h_1-k} \right), k \geq 0, Y(0) = 1, Y(1) \\ & = -1, Y(2) = 1 \end{aligned} \tag{18}$$

Utilizing the recurrence relation (18), we can see that

$$Y(3) = -\frac{0.4193733239}{\Gamma(3\alpha+1)},$$

$$Y(4) = -\frac{0.941528680\Gamma(\alpha+1)}{\Gamma(4\alpha+1)},$$

$$Y(5) = -\frac{0.2810318460\Gamma(2\alpha+1)}{\Gamma(5\alpha+1)}, \dots$$

By using the inverse differential transform of Y (k), which is;;

$$y(x) = \sum_{k=0}^{\infty} Y(k) x^{k\alpha}.$$

As a result, we arrive to the following series solution:

$$y(x) = \sum_{k=0}^{\infty} Y(k) x^{k\alpha}$$

$$= 1 - x^{\alpha} + x^{2\alpha} - \frac{0.4193733239}{\Gamma(3\alpha + 1)} x^{3\alpha}$$

$$- \frac{0.941528680\Gamma(\alpha + 1)}{\Gamma(4\alpha + 1)} x^{4\alpha} - \frac{0.2810318460\Gamma(2\alpha + 1)}{\Gamma(5\alpha + 1)} x^{5\alpha} + \cdots$$

$y(x) = \exp(-x)$ is exact solution, The graphs below show for various fractional values of α.

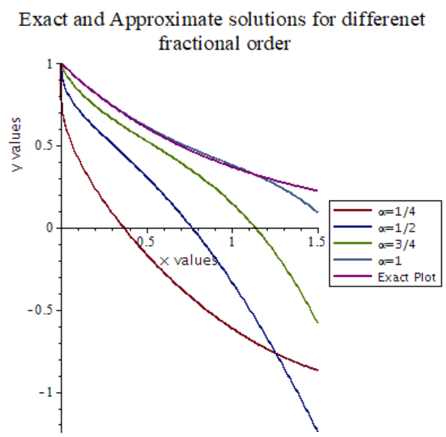

Fig. 8. For different fractional orders, a graph of exact and approximate solutions.

4. Conclusion

To solve fractional delay differential equations, the reduced differential transform method is utilized. It has been discovered that solving non-linear fractional differential equations with this method is quite simple and does not require the use of any discretization or Adomian polynomial. The resulting results converge to an exact answer. The exact and 10th iteration of the RDT approach are graphically compared. The proposed technique is found to be accurate and convergent for first-order and higher-order linear and non-linear problems.

Data availability statement

The data used to conduct this study is included in the paper.

Conflict of Interest

All the authors declare that they have no conflict of interest.

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