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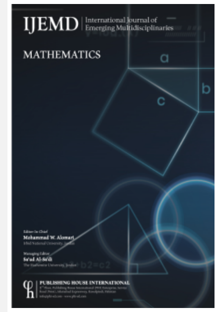

Homotopy Analysis Method for Solving System of Non-Linear Partial Differential Equations

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Abstract

This paper applies Homotopy Analysis Method (HAM) to obtain analytical solutions of system of non-linear partial differential equations. Numerical results clearly reflect complete compatibility of the proposed algorithm and discussed problems. Moreover, the validity of the present solution and suggested scheme is presented and the limiting case of presented findings is in excellent agreement with the available literature. The computed solution of the physical variables against the influential parameters is presented through graphs. Several examples are presented to show the efficiency and simplicity of the method.

Keywords: Homotopy Analysis Method; System of non-linear partial differential equations**.**

1. Introduction

Differential equations arise in almost all areas of the applied, physical and engineering sciences [1-29]. Recently [16-21], lot of attention is being paid on fractional differential equations and it has been observed that number of physical problems is better modeled by such equations. Several numerical and analytical techniques including Perturbation, Modified Adomian's Decomposition (MADM), Variational Iteration (VIM), Homotopy Perturbation (HPM) have been developed to solve such equations, see [16-21] and the references therein. Inspired and motivated by the ongoing research in this area, we apply Homotopy Analysis Method (HAM) [1-21] to obtain analytical solutions of non-linear system of partial differential equations.

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For analytical solution, solving nonlinear equations is more difficult than solving linear ones. Generally, there are two standards for satisfactory solutions. First, it can always provide analytical approximations efficiently and, second, it can give accurate enough analytical approximations for all pertinent parameters appearing in the governing expressions along with associated conditions. By using these two standard criteria, many numerical and analytical techniques are used to solve nonlinear equations. Among these, the homotopy analysis method (HAM) is one of the most powerful tools for solving nonlinear differential equations. Mostly, HAM is applied to boundary layer equations. Recently, the Homotopy analysis Method (HAM) was used by Marinca and Herisanu. [30]. Few relevant studies concerning the HAM are Refs. [31-32]. Numerical results are very encouraging and reveal the efficiency of proposed scheme (HAM).

1. Homotopy Analysis Method (HAM) [1-29]

We consider the following equation

$$\widetilde{N}[u(\tau)] = 0, \quad (1)$$

where $\widetilde{N}$ is a nonlinear operator, τ denotes dependent variables and $u(\tau)$ is an unknown function. For simplicity, we ignore all boundary and initial conditions, which can be treated in the similar way. By means of(HAM) Liao constructed zero-order deformation equation

$$(1-p)\mathcal{L}[\emptyset(\tau;p) - u_0(\tau)] = p\hbar\widetilde{N}[\emptyset(\tau;p)], \quad (2)$$

where $\mathcal{L}$ is a linear operator, $u_0(\tau)$ is an initial guess, $\hbar \neq 0$ is an auxiliary parameter and $p \in [0,1]$ is the embedding parameter. It is obvious that when p=0 and 1, it holds

$$\mathcal{L}[\emptyset(\tau;0) - u_0(\tau)] = 0 \quad \Longrightarrow \quad \emptyset(\tau;0) = u_0(\tau), \quad (3)$$

$$\hbar\widetilde{N}[\emptyset(\tau;1)] = 0 \Longrightarrow \emptyset(\tau;1) = u(\tau), \quad (4)$$

respectively. The solution $\emptyset(\tau;p)$ varies from initial guess $u_0(\tau)$ to solution $u(\tau)$. Liao [18] expanded $\emptyset(\tau;p)$ in Taylor series about the embedding parameter

$$\emptyset(\tau;p) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau)p^m, \quad (5)$$

where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \emptyset(\tau;p)}{\partial p^m}\Big|_{p=0} \quad (6)$$

The convergence of (5) depends on the auxiliary parameter $\hbar$. If this series is convergent at p=1,

$$\emptyset(\tau;1) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau), \tag{7}$$

Define vector

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), u_2(\tau), u_3(\tau), \dots\dots\dots, u_n(\tau)\}$$

If we differentiate the zeroth-order deformation equation Eq. (2) m-times with respect to p and then divide them $m!$ and finally set $p = 0$, we obtain the following mth-order deformation equation

$$\mathcal{L}[u_m(\tau) - X_m u_{m-1}(\tau)] = \hbar \Re_m(\vec{u}_{m-1}), \tag{8}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \widetilde{N}[\emptyset(\tau;p)]}{\partial p^{m-1}} \Big|_{p=0} \tag{9}$$

and

$$X_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1, \end{cases} \tag{10}$$

$If\ we\ multiply\ with\ L^{\wedge}(-1)\ each\ side\ of\ Eq.(8), we\ will\ obtain\ the\ following\ m$ th order deformation equation

$$u_m(\tau) = X_m u_{m-1}(\tau) + \hbar\, \Re_m\, (\vec{u}_{m-1})$$

2. Numerical Applications

In this section, we apply Homotopy Analysis Method (HAM) on the required problems. Numerical results are highly encouraging.

Example 1: consider the following system of nonlinear partial differential equations,

$$u_t + u_x v_x + u_y v_y + u = 0,$$

$$v_t + v_x w_x - v_y w_y - v = 0,$$

$$w_t + w_x u_x + w_y u_y - w = 0,$$

subject to the initial condition

$u(x, y, 0) = e^{x+y}$,

$v(x, y, 0) = e^{x-y}$,

$w(x, y, 0) = e^{-x+y}$,

To solve the above system of equations by HAM, the linear operator is defined as

$$L_1[U(x, y, t; q)] = \frac{\partial}{\partial t}[U(x, y, t; q)], \qquad L_1^{-1} = \int_0^t (.)dt,$$

$$L_2[V(x, y, t; q)] = \frac{\partial}{\partial t}[V(x, y, t; q)], \qquad L_2^{-1} = \int_0^t (.)dt,$$

$$L_3[W(x, y, t; q)] = \frac{\partial}{\partial t}[W(x, y, t; q)], \qquad L_3^{-1} = \int_0^t (.)dt,$$

With the property

$L_1[c_0] = 0$, $L_2[c_1] = 0$, $L_3[c_2] = 0$, where c_0, c_1, c_2 are integral constants and the non-linear operator is defined a

$$N_1[U(x, y, t; q)] = \frac{\partial}{\partial t}U(x, y, t; q) + \frac{\partial}{\partial x}U(x, y, t; q)\frac{\partial}{\partial x}V(x, y, t; q) + \frac{\partial}{\partial y}U(x, y, t; q)\frac{\partial}{\partial y}V(x, y, t; q) + U(x, y, t; q),$$

$$N_2[V(x, y, t; q)] = \frac{\partial}{\partial t}V(x, y, t; q) + \frac{\partial}{\partial x}V(x, y, t; q)\frac{\partial}{\partial x}W(x, y, t; q) - \frac{\partial}{\partial y}V(x, y, t; q)\frac{\partial}{\partial y}W(x, y, t; q) - V(x, y, t; q),$$

$$N_3[W(x, y, t; q)] = \frac{\partial}{\partial t}W(x, y, t; q) + \frac{\partial}{\partial x}W(x, y, t; q)\frac{\partial}{\partial x}U(x, y, t; q) + \frac{\partial}{\partial y}W(x, y, t; q)\frac{\partial}{\partial y}U(x, y, t; q) - W(x, y, t; q),$$

The zeroth order deformation is,

$$(1 - q)L_1\left[U(x, y, t; q) - u_0(x, y, t)\right] = qh_u N_1\, [U(x, y, t; q), V(x, y, t; q)], \qquad (11)$$

$$(1 - q)L_2[V(x, y, t; q) - v_0(x, y, t)] = qh_v N_2\, [V(x, y, t; q), W(x, y, t; q)], \qquad (12)$$

$$(1 - q)L_3[W(x, y, t; q) - w_0(x, y, t)] = qh_w N_2[W(x, y, t; q), U(x, y, t; q)], \qquad (13)$$

where $q \in [0,1]$ is an ebeding parameter, $h_u, h_v, h_w \neq 0$ are non-zero auxiliary parameters; $u_0(x,y,t), v_0(x,y,t), w_0(x,y,t)$ are initial guess.

for $q = 0$ $\quad U(x,y,t;0) = u_0(x,y,t),\ V(x,y,t;0) = v_0(x,y,t),\ W(x,y,t;0) = w_0(x,y,t),$

$q = 1$ $\quad U(x,y,t;1) = u(x,y,t),\ V(x,y,t;1) = v(x,y,t),\ W(x,y,t;1) = w(x,y,t),$

thus as q increses from 0 to 1 the solution $U(x,y,t;q), V(x,y,t;q), W(x,y,t;q)$ varies from the initial guess $u_0(x,y,t), v_0(x,y,t), w_0(x,y,t)$ to the solution $u(x,y,t), v(x,y,t), w(x,y,t)$. By Taylor series expansion, we have

$$U(x,y,t;q) = U(x,y,t;0) + q\,\frac{\partial}{\partial t}U(x,y,t;q) + q^2\frac{\partial^2}{\partial q^2}U(x,y,t;0) + q^3\frac{\partial^3}{\partial q^3}U(x,y,t;0) + \cdots,$$

$$= u_0(x,y,t) + \sum_{m=1}^{\infty} u_m\, q^m, \tag{14}$$

$$u_m = \frac{1}{m!}\frac{\partial^m U(x,y,t;q)}{\partial q^m}, \quad at\ q = 0,$$

$$V(x,y,t;q) = V(x,y,t;0) + q\,\frac{\partial}{\partial t}V(x,y,t;q) + q^2\frac{\partial^2}{\partial q^2}V(x,y,t;0) + q^3\frac{\partial^3}{\partial q^3}V(x,y,t;0) + \cdots,$$

$$= v_0(x,y,t) + \sum_{m=1}^{\infty} v_m\, q^m, \tag{15}$$

$$v_m = \frac{1}{m!}\frac{\partial^m V(x,y,t;q)}{\partial q^m}, \quad at\ q = 0,$$

$$W(x,y,t;q) = W(x,y,t;0) + q\,\frac{\partial}{\partial t}W(x,y,t;q) + q^2\frac{\partial^2}{\partial q^2}W(x,y,t;0) + q^3\frac{\partial^3}{\partial q^3}W(x,y,t;0) + \cdots,$$

$$= w_0(x,y,t) + \sum_{m=1}^{\infty} w_m\, q^m, \tag{16}$$

$$w_m = \frac{1}{m!}\frac{\partial^m W(x,y,t;q)}{\partial q^m}, \quad at\ q = 0,$$

For $q = 1$ eq (14), eq(15), eq(16) implies

$$u(x,y,t) = u_0(x,y,t) + \sum_{m=1}^{\infty} u_m(x,y,t),$$

$$v(x,y,t) = v_0(x,y,t) + \sum_{m=1}^{\infty} v_m(x,y,t),$$

$$w(x,y,t) = w_0(x,y,t) + \sum_{m=1}^{\infty} w_m(x,y,t),$$

Differentiating eq(11), eq(12), eq(13) wrt embedding parameter q, setting $q = 0$ and dividing by $m!$, the mth-order deformation is

$$L_1[u_m(x,y,t) - \chi_m u_{m-1}(x,y,t)] = h_u\, R_m(u_{m-1}),$$

$$where\ \ R_m(u_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N_1[U(x,y,t;q),V(x,y,t;q)}{\partial q^{m-1}}.$$

$$L_2[v_m(x,y,t) - \chi_m v_{m-1}(x,y,t)] = h_v\, R_m(u_{m-1}),$$

$$where\ \ R_m(v_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N_2[V(x,y,t;q),W(x,y,t;q)}{\partial q^{m-1}}.$$

$$L_3[w_m(x,y,t) - \chi_m w_{m-1}(x,y,t)] = h_w\, R_m(w_{m-1}),$$

$$where\ \ R_m(w_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N_3[W(x,y,t;q),U(x,y,t;q)}{\partial q^{m-1}}.$$

$$\chi_m = \begin{cases} 0\ m \le 1, \\ 1\ m < 1. \end{cases}$$

By taking $u_0(x,y,t) = e^{x+y}$, $v_0(x,y,t) = e^{x-y}$, $w_0(x,y,t) = e^{-x+y}$ and using the mth-order deformación. We have

$$u_1(x,y,t) = -e^{x+y}t\ ,$$

$$v_1(x,y,t) = e^{x-y}t\ ,$$

$$w_1(x,y,t) = e^{-x+y}t\ ,$$

$$u_2(x,y,t) = e^{x+y}\frac{t^2}{2!},$$

$$v_2(x,y,t) = e^{x-y}\frac{t^2}{2!},$$

$$w_2(x,y,t) = e^{-x+y}\frac{t^2}{2!},$$

$$u_3(x,y,t) = -e^{x+y}\frac{t^3}{3!},$$

$$v_3(x,y,t) = e^{x-y}\frac{t^3}{3!},$$

$$w_3(x,y,t) = e^{-x+y}\frac{t^3}{3!},$$

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The series solution is given by

$$u(x,y,t) = u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + \cdots,$$

$$v(x,y,t) = v_0(x,y,t) + v_1(x,y,t) + v_2(x,y,t) + \cdots,$$

$$w(x,y,t) = w_0(x,y,t) + w_1(x,y,t) + w_2(x,y,t) + \cdots,$$

$$u(x,y,t) = e^{x+y}\left[1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} - \cdots\right],$$

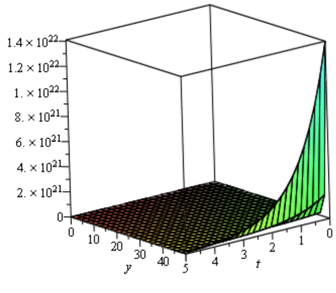
$$v(x,y,t) = e^{x-y}\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} - \cdots\right],$$

$$w(x,y,t) = e^{-x+y}\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} - \cdots\right],$$

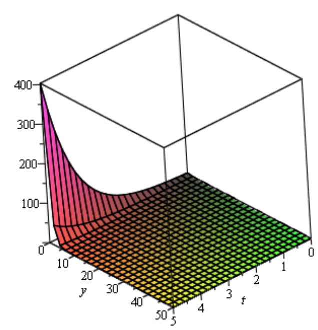
$$u(x,y,t) = e^{x+y-t},$$

$$v(x,y,t) = e^{x-y+t},$$

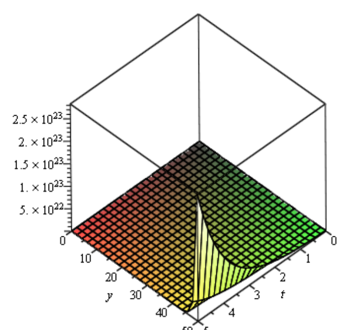
$$w(x,y,t) = e^{-x+y+t}.$$



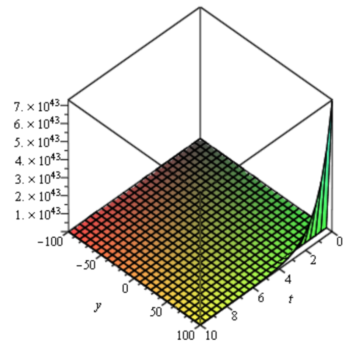
u at $t = 0..5, y = 0..50$



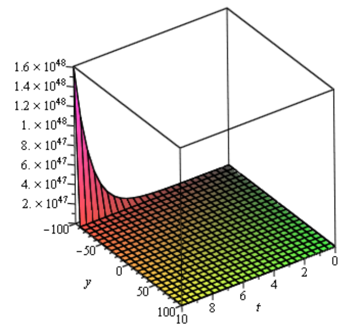
v at $t = 0..5, y = 0..50$



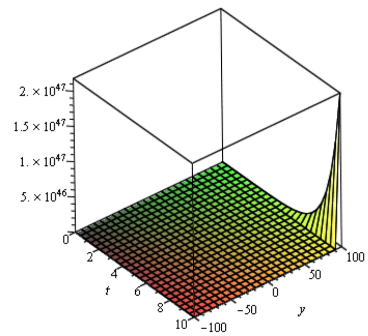
w at $t = 0..5, y = 0..50$



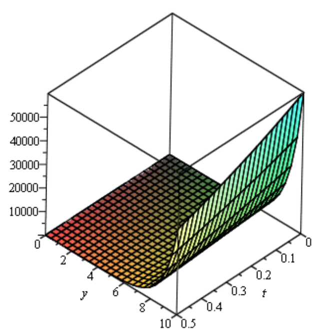
u at $t = 0..10, y = -100..100$



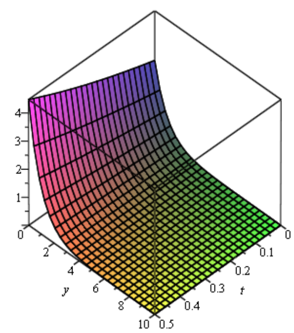
v at $t = 0..10, y = -100..100$



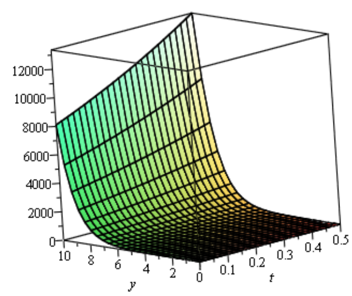
w at $t = 0..10, y = -100..100$



u at $t = 0..0.5, y = 0..10$



v at $t = 0..0.5, y = 0..10$



w at $t = 0..0.5, y = 0..10$

Example 2: consider the following system of nonlinear partial differential equations,

$u_t - v_x w_y = 1,$

$v_t - w_x u_y = 5,$

$w_t - u_x v_y = 5,$

subject to the initial ondition

$u(x, y, 0) = x + 2y,$

$v(x, y, 0) = x - 2y,$

$w(x, y, 0) = -x + 2y,$

To solve the above system of equations by HAM, the linear operator is defined as

$$L_1[U(x,y,t;q)] = \frac{\partial}{\partial t}[U(x,y,t;q)], \qquad L_1{}^{-1} = \int_0^t (.)dt,$$

$$L_2[V(x,y,t;q)] = \frac{\partial}{\partial t}[V(x,y,t;q)], \qquad L_2{}^{-1} = \int_0^t (.)dt,$$

$$L_3[W(x,y,t;q)] = \frac{\partial}{\partial t}[W(x,y,t;q)], \qquad L_3{}^{-1} = \int_0^t (.)dt,$$

With the property

$L_1[c_0] = 0$, $L_2[c_1] = 0$, $L_3[c_2] = 0$, where c_0, c_1, c_2 are integral constants

and the non-linear operator is defined as,

$$N_1[U(x,y,t;q)] = \frac{\partial}{\partial t}U(x,y,t;q) - \frac{\partial}{\partial x}V(x,y,t;q)\frac{\partial}{\partial y}W(x,y,t;q) - 1,$$

$$N_2[V(x,y,t;q)] = \frac{\partial}{\partial t}V(x,y,t;q) - \frac{\partial}{\partial x}W(x,y,t;q)\frac{\partial}{\partial y}U(x,y,t;q) - 5,$$

$$N_3[W(x,y,t;q)] = \frac{\partial}{\partial t}W(x,y,t;q) - \frac{\partial}{\partial x}U(x,y,t;q)\frac{\partial}{\partial y}V(x,y,t;q) - 5,$$

The zeroth order deformation is,

$$(1-q)L_1\big[U(x,y,t;q) — u_0(x,y,t)\big] = qh_u N_1\,[U(x,y,t;q), V(x,y,t;q), W(x,y,t;q)], \tag{17}$$

$$(1-q)L_2[V(x,y,t;q) — v_0(x,y,t)] = qh_v N_2\,[V(x,y,t;q), W(x,y,t;q), U(x,y,t;q)], \tag{18}$$

$$(1-q)L_3[W(x,y,t;q) — w_0(x,y,t)] = qh_w N_2[W(x,y,t;q), U(x,y,t;q), V(x,y,t;q)], \tag{19}$$

where $q \in [\,0\,,1\,]$ is an ebeding parameter,$h_u, h_v, h_w \neq 0$ are non-zero auxiliary parameters; $u_0(x,y,t), v_0(x,y,t), w_0(x,y,t)$ are initial guess.

for $q = 0$ $U(\mathrm{x,y,t};0) = u_0(\mathrm{x,y,t})$, $V(\mathrm{x,y,t};0) = v_0(\mathrm{x,y,t})$, $W(\mathrm{x,y,t};0) = w_0(\mathrm{x,y,t})$,

$q = 1$ $U(\mathrm{x,y,t};1) = u(\mathrm{x,y,t})$, $V(\mathrm{x,y,t};1) = v(\mathrm{x,y,t})$, $W(\mathrm{x,y,t};1) = w(\mathrm{x,y,t})$, thus as q increses from 0 to 1 the solution $U(x,y,t;q), V(x,y,t;q), W(x,y,t;q)$ varies from the initial guess $u_0(x,y,t), v_0(x,y,t), w_0(x,y,t)$ to the solution $u(x,y,t), v(x,y,t), w(x,y,t)$. By Taylor series expansion, we have

$$U(x,y,t;q) = U(x,y,t;0) + q\frac{\partial}{\partial t}U(x,y,t;q) + q^2\frac{\partial^2}{\partial q^2}U(x,y,t;0) + q^3\frac{\partial^3}{\partial q^3}U(x,y,t;0) + \cdots,$$

$$= u_0(x,y,t) + \sum_{m=1}^{\infty} u_m q^m, \tag{20}$$

$$u_m = \frac{1}{m!}\frac{\partial^m U(x,y,t;q)}{\partial q^m}, \quad at\ q = 0,$$

$$V(x,y,t;q) = V(x,y,t;0) + q\frac{\partial}{\partial t}V(x,y,t;q) + q^2\frac{\partial^2}{\partial q^2}V(x,y,t;0) + q^3\frac{\partial^3}{\partial q^3}V(x,y,t;0) + \cdots,$$

$$= v_0(x,y,t) + \sum_{m=1}^{\infty} v_m q^m, \tag{21}$$

$$v_m = \frac{1}{m!}\frac{\partial^m V(x,y,t;q)}{\partial q^m}, \quad at\ q = 0,$$

$$W(x,y,t;q) = W(x,y,t;0) + q\frac{\partial}{\partial t}W(x,y,t;q) + q^2\frac{\partial^2}{\partial q^2}W(x,y,t;0) + q^3\frac{\partial^3}{\partial q^3}W(x,y,t;0) + \cdots,$$

$$= w_0(x,y,t) + \sum_{m=1}^{\infty} w_m q^m, \tag{22}$$

$$w_m = \frac{1}{m!}\frac{\partial^m W(x,y,t;q)}{\partial q^m}, \quad at\ q = 0,$$

For $q = 1$ eq (5), eq(6), eq(7) implies,

$$u(x,y,t) = u_0(x,y,t) + \sum_{m=1}^{\infty} u_m(x,y,t),$$

$$v(x,y,t) = v_0(x,y,t) + \sum_{m=1}^{\infty} v_m(x,y,t),$$

$$w(x,y,t) = w_0(x,y,t) + \sum_{m=1}^{\infty} w_m(x,y,t),$$

Differentiating eq(17),eq(18),eq(19) wrt embedding parameter q, setting $q = 0$ and dividing by $m!$, the mth-order deformation is

$$L_1[u_m(x,y,t) - \chi_m u_{m-1}(x,y,t)] = h_u\, R_m(u_{m-1}),$$

$$where\ R_m(u_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N_1[U(x,y,t;q), V(x,y,t;q), W(x,y,t;q)}{\partial q^{m-1}}.$$

$$L_2[v_m(x,y,t) - \chi_m v_{m-1}(x,y,t)] = h_v\, R_m(u_{m-1}),$$

$$where\ \ R_m(v_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N_2[V(x,y,t;q),W(x,y,t;q),U(x,y,t;q)}{\partial q^{m-1}}.$$

$L_3[w_m(x,y,t) - \chi_m w_{m-1}(x,y,t)] = h_w\ R_m(w_{m-1})$,

$$where\ \ R_m(w_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N_3[W(x,y,t;q),U(x,y,t;q),V(x,y,t;q)}{\partial q^{m-1}}.$$

$$\chi_m = \begin{cases} 0\ m \le 1, \\ 1\ m < 1. \end{cases}$$

By taking $u_0(x,y,t) = x + 2y$, $v_0(x,y,t) = x - 2y$, $w_0(x,y,t) = -x + 2y$ and using the mth-order deformation,we have

$u_1(x,y,t) = 3t$,

$v_1(x,y,t) = 3t$,

$w_1(x,y,t) = 3t$,

$u_k(x,y,t) = 0\, , for\ k \ge 2$

$v_k(x,y,t) = 0\, , for\ k \ge 2$

$w_k(x,y,t) = 0\, , for\ k \ge$

.

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The series solution is given by

$u(x,y,t) = u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + \cdots$,

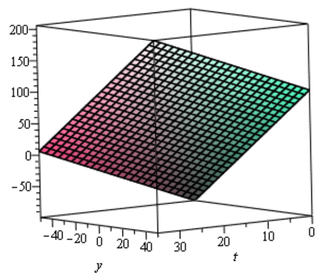
$v(x,y,t) = v_0(x,y,t) + v_1(x,y,t) + v_2(x,y,t) + \cdots$,

$w(x,y,t) = w_0(x,y,t) + w_1(x,y,t) + w_2(x,y,t) + \cdots$,

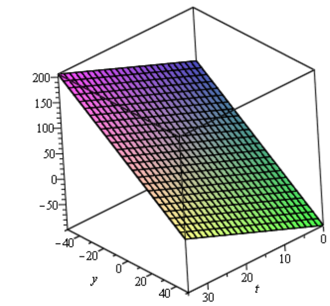
$u(x,y,t) = x + 2y + 3t$,

$$v(x, y, t) = x - 2y + 3t,$$

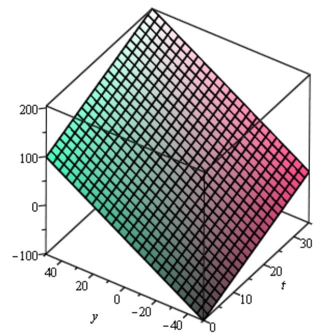
$$w(x, y, t) = -x + 2y + 3t,$$



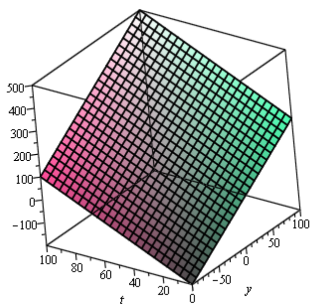
u at t = 0 ..35, *y* =−50 ..50



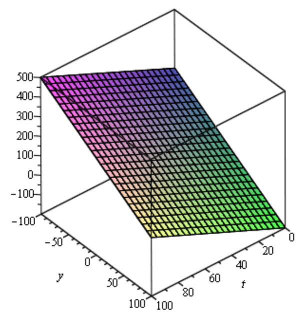
v at t = 0 ..35, *y* =−50 ..50



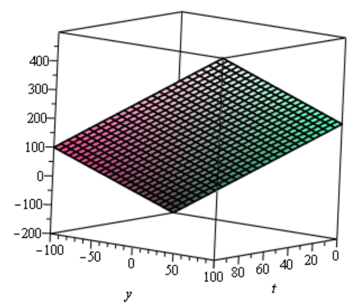
w at t = 0 ..35, *y* =−50 ..50



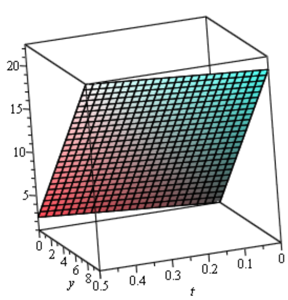
u at t = 0 ..100, *y* =−100 ..100



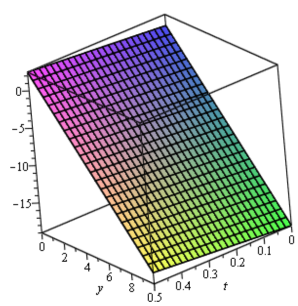
v at t = 0 ..100, *y* =−100 ..100



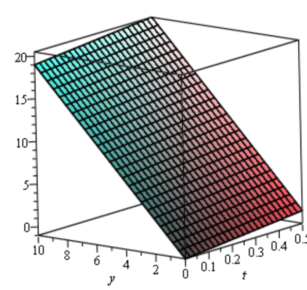
w at t = 0 ..100, *y* =−100 ..100



u at t = 0 ..0.5, *y* = 0 ..10



v at t = 0 ..0.5, *y* = 0 ..10



w at t = 0 ..0.5, *y* = 0 ..10

3. Conclusions

Homotopy Analysis Method (HAM) is implemented to obtain analytical solutions of non-linear system of partial differential equations. Comparison of the obtained results HAM with exact solution shows that the method is reliable and capable of providing analytic treatment for solving such equations. Numerical results and graphical representations clearly reflect complete compatibility of the proposed algorithm and discussed problems.

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