

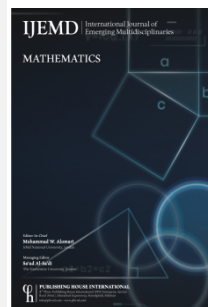


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Homotopy Analysis Method for Non-Linear Schrödinger Equations

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Abstract

This paper applies Homotopy Analysis Method (HAM) to obtain analytical solutions of nonlinear Schrödinger equations. Numerical results clearly reflect complete compatibility of the proposed algorithm and discussed problems. Several examples are presented to show the efficiency and simplicity of the method.

Keywords: Homotopy Analysis Method; fractional calculus and nonlinear Schrödinger equations.

1. Introduction

Differential equations arise in almost all areas of the applied, physical and engineering sciences [1-21]. Recently [16-21], lot of attention is being paid on fractional differential equations and it has been observed that number of physical problems is better modeled by such equations. Several numerical and analytical techniques including Perturbation, Modified Adomian's Decomposition (MADM), Variational Iteration (VIM), Homotopy Perturbation (HPM) have been developed to solve such equations, see [16-21] and the references therein. Inspired and motivated by the ongoing research in this area, we apply Homotopy Analysis Method (HAM) [1-21] to obtain analytical solutions of nonlinear Schrödinger equations. In particular, the Schrödinger equation occurs in the various areas of physics, including nonlinear optics, plasma physics, super conductivity and quantum mechanics. The non-linear Schrödinger equation

exhibits solitary type solution. Many methods are usually used to handle the nonlinear equation such as inverse scattering methods, tanh method, Backlund transformation and others methods as well.

Consider the Schrödinger equation with the following initial condition

$$i \frac{\partial \psi(X, t)}{\partial t} = -\frac{1}{2} \nabla^2 \psi + V_d(X) \psi + \beta_d |\psi|^2 \psi, \quad X \in \mathbb{R}^d, t \geq 0$$

$$\psi(X, 0) = \psi^0(X), \quad X \in \mathbb{R}^d,$$

where $V_d(X)$ is the trapping potential and β_d is a real constant. Numerical results are very encouraging and reveal the efficiency of proposed scheme (HAM).

2. Homotopy Analysis Method (HAM) [1-21]

We consider the following equation

$$\tilde{N}[u(\tau)] = 0, \quad (1)$$

where \tilde{N} is a nonlinear operator, τ denotes dependent variables and $u(\tau)$ is an unknown function. For simplicity, we ignore all boundary and initial conditions, which can be treated in the similar way. By means of HAM L [1-21] constructed zero-order deformation equation

$$(1 - p)\mathcal{L}[\phi(\tau; p) - u_0(\tau)] = p\hbar\tilde{N}[\phi(\tau; p)], \quad (2)$$

Where \mathcal{L} is a linear operator, $u_0(\tau)$ is an initial guess, $\hbar \neq 0$ is an auxiliary parameter and $p \in [0, 1]$ is the embedding parameter. It is obvious that when $p=0$ and 1, it holds

$$\mathcal{L}[\phi(\tau; 0) - u_0(\tau)] = 0 \implies \phi(\tau; 0) = u_0(\tau), \quad (3)$$

$$\hbar\tilde{N}[\phi(\tau; 1)] = 0 \implies \phi(\tau; 1) = u(\tau), \quad (4)$$

respectively. The solution $\phi(\tau; p)$ varies from initial guess $u_0(\tau)$ to solution $u(\tau)$. Liao [18] expanded $\phi(\tau; p)$ in Taylor series about the embedding parameter

$$\phi(\tau; p) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau) p^m, \quad (5)$$

where

$$u_m(\tau) = \frac{1}{m!} \left. \frac{\partial^m \phi(\tau; p)}{\partial p^m} \right|_{p=0} \quad (6)$$

The convergence of (5) depends on the auxiliary parameter \hbar . If this series is convergent at $p=1$,

$$\varphi(\tau; 1) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau), \quad (7)$$

Define vector

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), u_2(\tau), u_3(\tau), \dots, \dots, u_n(\tau)\}$$

If we differentiate the zeroth-order deformation equation Eq. (2) m -times with respect to p and then divide them $m!$ and finally set $p = 0$, we obtain the following m th-order deformation equation

$$\mathcal{L}[u_m(\tau) - X_m u_{m-1}(\tau)] = \hbar \mathfrak{R}_m(\vec{u}_{m-1}), \quad (8)$$

where

$$\mathfrak{R}_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \tilde{N}[\varphi(\tau; p)]}{\partial p^{m-1}} \right|_{p=0} \quad (9)$$

and

$$X_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1, \end{cases} \quad (10)$$

If we multiply with \mathcal{L}^{-1} each side of Eq. (8), we will obtain the following m th order deformation equation

$$u_m(\tau) = X_m u_{m-1}(\tau) + \hbar \mathfrak{R}_m(\vec{u}_{m-1})$$

3. Numerical Applications

In this section, we apply Homotopy Analysis Method (HAM) on the required problems. Numerical results are highly encouraging.

Example 1: Consider the following one dimensional Schrödinger equation

$$i \frac{\partial \psi(x, t)}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} - |\psi|^2 \psi, \quad t \geq 0, \quad (11)$$

subject to the initial condition

$$\psi(x, 0) = e^{ix}, \quad t \geq 0,$$

To solve equation (1) by HAM, the linear operator is defined as

$$L[\varphi(x, t; q)] = \frac{\partial^\alpha}{\partial t^\alpha} [\varphi(x, t; q)], \quad 0 < \alpha \leq 1, \quad L^{-1} = J_*^\alpha(\cdot),$$

With the property $L[C] = 0$, where C is constant of integration

and the non-linear operator is defined as

$$N[\varphi(x, t; q)] = -\frac{\partial}{\partial t} \varphi(x, t; q) + \frac{i}{2} \frac{\partial^2}{\partial x^2} \varphi(x, t; q) + i |\varphi(x, t; q)|^2 \varphi(x, t; q).$$

The zeroth order deformation is,

$$(1 - q)L[\varphi(x, t; q) - \psi_0(x, t)] = qhN[\varphi(x, t; q)]. \quad (12)$$

where $q \in [0, 1]$ is an embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter; $\psi_0(x, t)$ is initial guess.

$$\text{for } q = 0, \quad \varphi(x, t; 0) = \psi_0(x, t),$$

$$\text{for } q = 1, \quad \varphi(x, t; 1) = \psi(x, t),$$

thus as q increases from 0 to 1 the solution $\varphi(x, t; q)$ varies from the initial guess $\psi_0(x, t)$ to the solution $\psi(x, t)$. By Taylor series expansion, we have

$$\begin{aligned} \varphi(x, t; q) &= \varphi(x, t; 0) + q \frac{\partial}{\partial q} \varphi(x, t; q) + q^2 \frac{\partial^2}{\partial q^2} \varphi(x, t; 0) + q^3 \frac{\partial^3}{\partial q^3} \varphi(x, t; 0) + \dots, \\ &= \psi_0(x, t) + \sum_{m=1}^{\infty} \psi_m q^m, \end{aligned} \quad (13)$$

$$\psi_m = \frac{1}{m!} \frac{\partial^m \varphi(x, t; q)}{\partial q^m}, \quad \text{at } q = 0,$$

For $q = 1$ eq (3) implies

$$\psi(x, t) = \psi_0(x, t) + \sum_{m=1}^{\infty} \psi_m(x, t).$$

Differentiating (2) with respect to embedding parameter q , setting $q = 0$ and dividing by $m!$, the m th-order deformation is

$$L[\psi_m(x, t) - \chi_m \psi_{m-1}(x, t)] = h R_m(\psi_{m-1}),$$

$$\text{where } R_m(\psi_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\varphi(x, t; q)]}{\partial q^{m-1}}.$$

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases}$$

By taking $\psi_0(x, t) = e^{ix}$, differentiate eq (2) with respect to q and setting $q = 0$, we have

$$L\left[\frac{\partial}{\partial q}\varphi(x, t; 0)\right] = hN[\varphi(x, t; 0)],$$

$$L[\psi_1(x, t)] = h\left[-\frac{\partial}{\partial t}\varphi(x, t; 0) + \frac{i}{2}\frac{\partial^2}{\partial x^2}\varphi(x, t; 0) + i|\varphi(x, t; 0)|^2\varphi(x, t; 0)\right],$$

$$L[\psi_1(x, t)] = h\left[-\frac{\partial}{\partial t}e^{ix} + \frac{i}{2}\frac{\partial^2}{\partial x^2}e^{ix} + i|e^{ix}|^2e^{ix}\right],$$

$$\psi_1(x, t) = hL^{-1}\left[\frac{i}{2}e^{ix}\right],$$

$$= hJ^\alpha\left[\frac{i}{2}e^{ix}\right],$$

$$= h\frac{i}{2}e^{ix}\left[\frac{t^\alpha}{\Gamma(\alpha+1)}\right],$$

$$\psi_2(x, t) = -\frac{1}{4}h^2e^{ix}\left[\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right],$$

$$\psi_3(x, t) = -\frac{1}{8}h^3ie^{ix}\left[\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right],$$

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The series solution is given by

$$\psi(x, t) = \psi_0(x, t) + \psi_1(x, t) + \psi_2(x, t) + \psi_3(x, t) + \dots,$$

$$\psi(x, t) = e^{ix} + h\frac{i}{2}e^{ix}\left[\frac{t^\alpha}{\Gamma(\alpha+1)}\right] + -\frac{1}{4}h^2e^{ix}\left[\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right] + -\frac{1}{8}h^3ie^{ix}\left[\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right] \dots,$$

For $\alpha = 1$ and $h = 1$, we have

$$\psi(x, t) = e^{ix}\left[1 + \frac{1}{2}it + \frac{1}{2!}\left(\frac{1}{2}it\right)^2 + \frac{1}{3!}\left(\frac{1}{2}it\right)^3 + \dots\right],$$

$$= e^{ix}e^{\frac{it}{2}},$$

$$= e^{i\left(x+\frac{t}{2}\right)},$$

which is the exact solution.

Real part

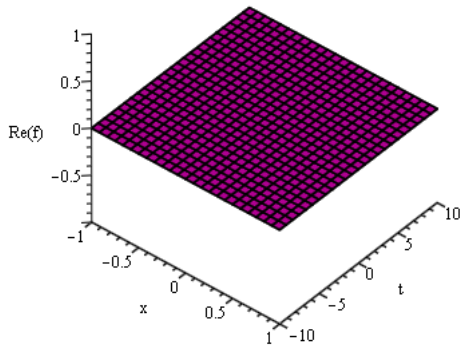


Figure 1 :

For $-1 \leq x \leq 1, -10 \leq t \leq 10$

Real part

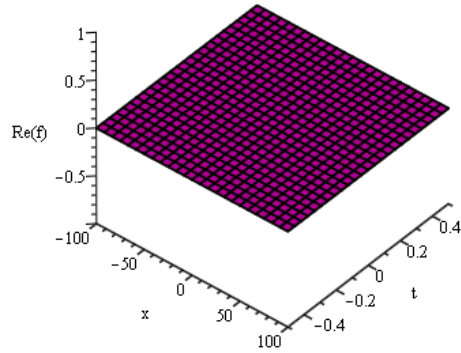


Figure 2:

For $-100 \leq x \leq 100, -0.5 \leq t \leq 0.5$

Imaginary part

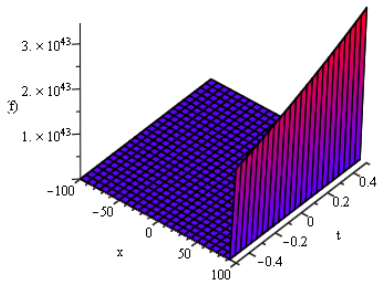


Figure 3:

For $-100 \leq x \leq 100, -0.5 \leq t \leq 0.5$

Imaginary part

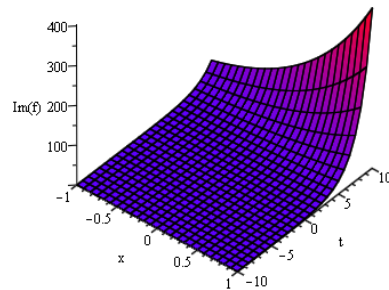


Figure 4:

For $-1 \leq x \leq 1, -10 \leq t \leq 10$

Example 2: Consider the following one dimensional Schrödinger equation

$$i \frac{\partial \psi(x, t)}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + \psi \cos^2 x + |\psi|^2 \psi, \quad t \geq 0, \quad (14)$$

subject to the initial condition

$$\psi(x, 0) = \sin x,$$

To solve equation (1) by HAM, the linear operator is defined as

$$L[\varphi(x, t; q)] = \frac{\partial^\alpha}{\partial t^\alpha} [\varphi(x, t; q)], \quad 0 < \alpha \leq 1, \quad L^{-1} = J_*^\alpha(\cdot),$$

With the property $L[C] = 0$, where C is constant of integration

and the non-linear operator is defined as

$$N[\varphi(x, t; q)] = -\frac{\partial}{\partial t} \varphi(x, t; q) + \frac{i}{2} \frac{\partial^2}{\partial x^2} \varphi(x, t; q) - i\varphi(x, t; q) \cos^2 x - i|\varphi(x, t; q)|^2 \varphi(x, t; q).$$

The zeroth order deformation is,

$$(1 - q)L[\varphi(x, t; q) - \psi_0(x, t)] = qhN[\varphi(x, t; q)]. \quad (15)$$

where $q \in [0, 1]$ is an embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter; $\psi_0(x, t)$ is initial guess.

$$\text{for } q = 0, \quad \varphi(x, t; 0) = \psi_0(x, t),$$

$$\text{for } q = 1, \quad \varphi(x, t; 1) = \psi(x, t),$$

thus as q increases from 0 to 1 the solution $\psi(x, t; q)$ varies from the initial guess $\psi_0(x, t)$ to the solution $\psi(x, t)$. By Taylor series expansion, we have

$$\begin{aligned} \varphi(x, t; q) &= \varphi(x, t; 0) + q \frac{\partial}{\partial t} \varphi(x, t; 0) + q^2 \frac{\partial^2}{\partial q^2} \varphi(x, t; 0) + q^3 \frac{\partial^3}{\partial q^3} \varphi(x, t; 0) + \dots, \\ &= \psi_0(x, t) + \sum_{m=1}^{\infty} \psi_m q^m, \end{aligned} \quad (16)$$

$$\psi_m = \frac{1}{m!} \frac{\partial^m \varphi(x, t; q)}{\partial q^m}, \quad \text{at } q = 0,$$

For $q = 1$ eq (3) implies

$$\psi(x, t) = \psi_0(x, t) + \sum_{m=1}^{\infty} \psi_m(x, t).$$

Differentiating (2) w.r.t embedding parameter q , setting $q = 0$ and dividing by $m!$, the m th-order deformation is

$$L[\psi_m(x, t) - \chi_m \psi_{m-1}(x, t)] = h R_m(\psi_{m-1}),$$

$$\text{where } R_m(\psi_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\varphi(x, t; q)]}{\partial q^{m-1}}.$$

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m < 1. \end{cases}$$

By taking $\psi_0(x, t = \sin x)$, differentiating (2) w.r.t q and setting $q = 0$, we have

$$L\left[\frac{\partial}{\partial q}\varphi(x, t; 0)\right] = hN[\varphi(x, t; 0)],$$

$$L[\psi_1(x, t)] = h\left[-\frac{\partial}{\partial t}\varphi(x, t; 0) + \frac{i}{2}\frac{\partial^2}{\partial x^2}\varphi(x, t; 0) - i\varphi(x, t; 0)\cos^2x + i|\varphi(x, t; 0)|^2\varphi(x, t; 0)\right],$$

$$L[\psi_1(x, t)] = h\left[-\frac{\partial}{\partial t}\sin x + \frac{i}{2}\frac{\partial^2}{\partial x^2}\sin x - i\sin x\cos^2x + i|\sin x|^2\sin x\right],$$

$$\psi_1(x, t) = hL^{-1}\left[-\frac{3i}{2}\sin x\right],$$

$$= hJ^\alpha\left[-\frac{3i}{2}\sin x\right],$$

$$= h\left[-\frac{3i}{2}\sin x\frac{t^\alpha}{\Gamma(\alpha+1)}\right],$$

$$\psi_2(x, t) = h^2\left[-\frac{9}{4}\sin x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right],$$

$$\psi_3(x, t) = h^3\left[\frac{27}{8}i\sin x\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right],$$

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The series solution is given by

$$\psi(x, t) = \psi_0(x, t) + \psi_1(x, t) + \psi_2(x, t) + \psi_3(x, t) + \dots,$$

$$\psi(x, t) = \sin x\left[1 + h\left[-\frac{3i}{2}\frac{t^\alpha}{\Gamma(\alpha+1)}\right] + h^2\left[-\frac{9}{4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right] + h^3\left[\frac{27}{8}i\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right] + \dots\right],$$

For $\alpha = 1$ and $h = 1$, we have

$$\psi(x, t) = \sin x\left[1 + \left(-\frac{3}{2}it\right) + \frac{1}{2!}\left(-\frac{3}{2}it\right)^2 + \frac{1}{3!}\left(-\frac{3}{2}it\right)^3 + \dots\right],$$

$$= \sin x e^{\frac{-3it}{2}},$$

which is the exact solution.

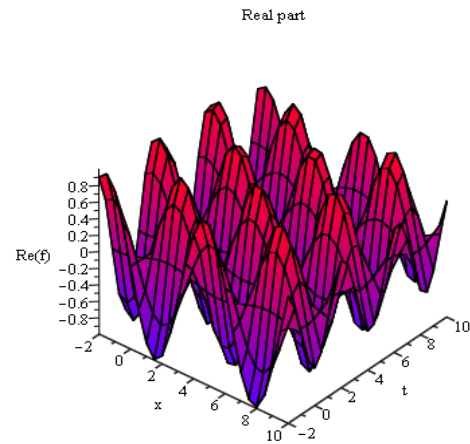


Figure 5: For $-2 \leq x \leq 10$, $-2 \leq t \leq 10$

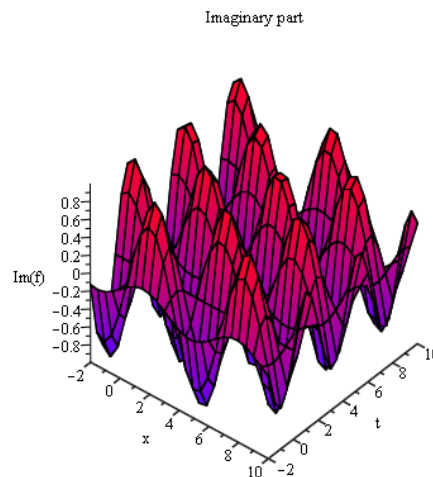


Figure 6: For $-2 \leq x \leq 10$, $-2 \leq t \leq 10$,

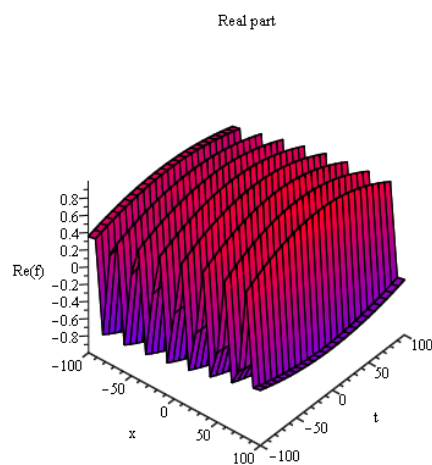


Figure 7: For $-100 \leq x \leq 100$, $-100 \leq t \leq 100$,

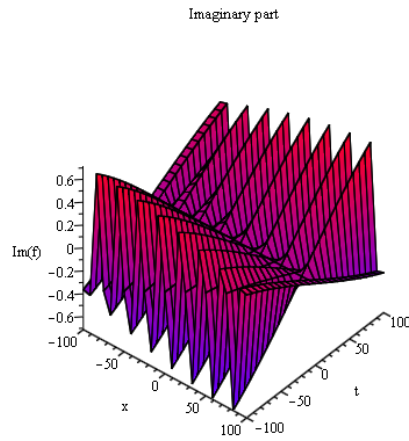


Figure 7: For $-100 \leq x \leq 100$, $-100 \leq t \leq 100$,

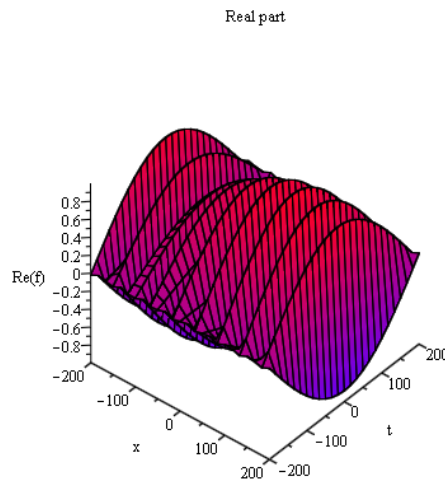
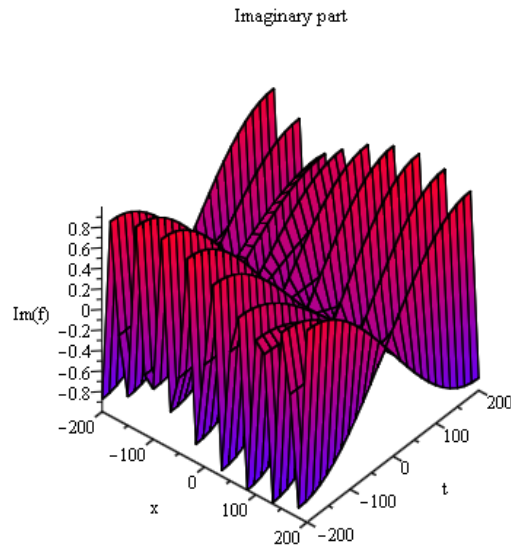


Figure 8: For $-200 \leq x \leq 200$, $-200 \leq t \leq 200$,



for $-200 \leq x \leq 200$, $-200 \leq t \leq 200$,

Example 3: Consider the following two dimensional Schrödinger equation

$$i \frac{\partial \psi(x, y, t)}{\partial t} = -\frac{1}{2} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) + V(x, y) \psi + |\psi|^2 \psi, \quad (x, y) \in [0, 2\pi] \times [0, 2\pi],$$

subject to the initial condition

$$\psi(x, y, 0) = \sin x \sin y,$$

$$\text{where } V(x, y) = 1 - \sin^2 x \sin^2 y,$$

To solve equation (1) by HAM, the linear operator is defined as

$$L[\varphi(x, y, t; q)] = \frac{\partial^\alpha}{\partial t^\alpha} [\varphi(x, y, t; q)], \quad 0 < \alpha \leq 1, \quad L^{-1} = J_*^\alpha(\cdot),$$

With the property $L[C] = 0$, where C is constant of integration

and the non-linear operator is defined as

$$\begin{aligned} N(x, y, t; q) = & \\ & -\frac{\partial}{\partial t} \varphi(x, y, t; q) + \frac{i}{2} \left(\frac{\partial^2}{\partial x^2} \varphi(x, y, t; q) + \frac{\partial^2}{\partial y^2} \varphi(x, y, t; q) \right) - i \varphi(x, y, t; q) V(x, y) - \\ & i |\varphi(x, y, t; q)|^2 \varphi(x, y, t; q). \end{aligned}$$

The zeroth order deformation is,

$$(1 - q)L[\varphi(x, y, t; q) - \psi_0(x, y, t)] = qhN[\varphi(x, y, t; q)]. \quad (17)$$

where $q \in [0, 1]$ is an embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter; $\psi_0(x, y, t)$ is initial guess.

$$\text{for } q = 0, \quad \varphi(x, y, t; 0) = \psi_0(x, y, t),$$

$$\text{for } q = 1, \quad \varphi(x, y, t; 1) = \psi(x, y, t),$$

thus as q increases from 0 to 1 the solution $\varphi(x, y, t; q)$ varies from the initial guess $\psi_0(x, y, t)$ to the solution $\psi(x, y, t)$.

By Taylor's series expansion, we have

$$\begin{aligned} \varphi(x, y, t; q) &= \varphi(x, y, t; 0) + q \frac{\partial}{\partial q} \varphi(x, y, t; q) + q^2 \frac{\partial^2}{\partial q^2} \varphi(x, y, t; 0) + q^3 \frac{\partial^3}{\partial q^3} \varphi(x, y, t; 0) + \dots, \\ &= \psi_0(x, y, t) + \sum_{m=1}^{\infty} \psi_m q^m, \end{aligned} \quad (18)$$

$$\psi_m = \frac{1}{m!} \frac{\partial^m \varphi(x, y, t; q)}{\partial q^m}, \quad \text{at } q = 0,$$

For $q = 1$ eq (3) implies

$$\psi(x, y, t) = \psi_0(x, y, t) + \sum_{m=1}^{\infty} \psi_m(x, y, t).$$

Differentiating (2) wrt embedding parameter q , setting $q = 0$ and dividing by $!$, the m th-order deformation is

$$L[\psi_m(x, y, t) - \chi_m \psi_{m-1}(x, y, t)] = h R_m(\psi_{m-1}),$$

$$\text{where } R_m(\psi_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\varphi(x, y, t; q)]}{\partial q^{m-1}}.$$

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m < 1. \end{cases}$$

By taking $\psi_0(x, t = \sin x \sin y)$, differentiating (2) w.r.t q and setting $q = 0$, we have

$$L\left[\frac{\partial}{\partial q} \varphi(x, y, t; 0)\right] = h N[\varphi(x, y, t; 0)],$$

$$L[\psi_1(x, t)] = h \left[-\frac{\partial}{\partial t} \varphi(x, y, t; 0) + \frac{i}{2} \left(\frac{\partial^2}{\partial x^2} \varphi(x, y, t; 0) + \frac{\partial^2}{\partial y^2} \varphi(x, y, t; 0) \right) - i \varphi(x, y, t; 0) (1 - \sin^2 x \sin^2 y) \right. \\ \left. + i |\varphi(x, y, t; 0)|^2 \varphi(x, y, t; 0) \right],$$

$$L[\psi_1(x, y, t)] = h \left[-\frac{\partial}{\partial t} \sin x \sin y + \frac{i}{2} \left(\frac{\partial^2}{\partial x^2} \sin x \sin y + \frac{\partial^2}{\partial y^2} \sin x \sin y \right) - i \sin x \sin y (1 - \sin^2 x \sin^2 y) \right. \\ \left. + i |\sin x \sin y|^2 \sin x \sin y \right],$$

$$\psi_1(x, y, t) = h L^{-1}[-2i \sin x \sin y],$$

$$= h J^\alpha [-2i \sin x \sin y],$$

$$= h \left[-2i \sin x \sin y \frac{t^\alpha}{\Gamma(\alpha+1)} \right],$$

$$\psi_2(x, y, t) = h^2 \left[-4 \sin x \sin y \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right],$$

$$\psi_3(x, y, t) = h^3 \left[8i \sin x \sin y \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \right],$$

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The series solution is given by

$$\psi(x, y, t) = \psi_0(x, y, t) + \psi_1(x, y, t) + \psi_2(x, y, t) + \psi_3(x, y, t) + \dots,$$

$$\psi(x, y, t) = \sin x \sin y \left[1 + h \left[-2i \frac{t^\alpha}{\Gamma(\alpha+1)} \right] + h^2 \left[-4 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right] + h^3 \left[8i \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \right] \right] + \dots,$$

For $\alpha = 1$ and $h = 1$, we have

$$\psi(x, y, t) = \sin x \sin y \left[1 + (-2it) + \frac{1}{2!} (-2it)^2 + \frac{1}{3!} (-2it)^3 + \dots \right],$$

$$= \sin x \sin y e^{-2it},$$

which is exact solution.

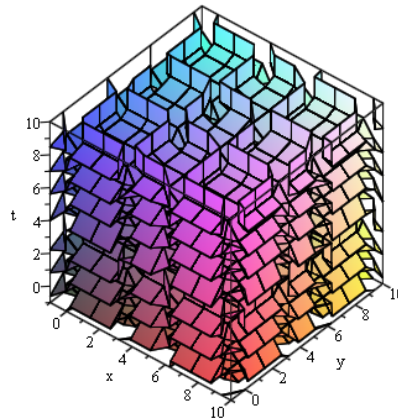


Figure 9: Real part for $-1 \leq x \leq 10$, $-1 \leq t \leq 10$,

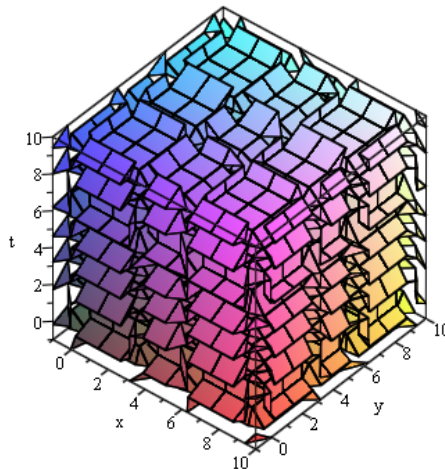


Figure 10: For imaginary part $-1 \leq x \leq 10$, $-1 \leq t \leq 10$,

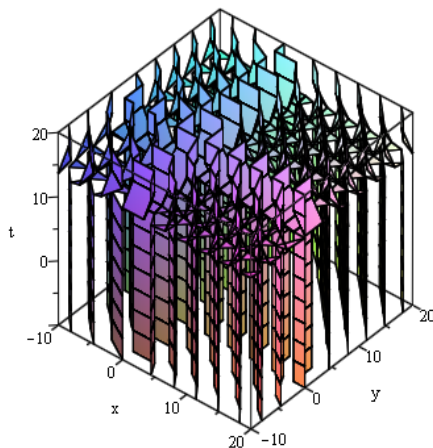


Figure 11: For real part $-10 \leq x \leq 20, -10 \leq y \leq 20, -10 \leq t \leq 20$

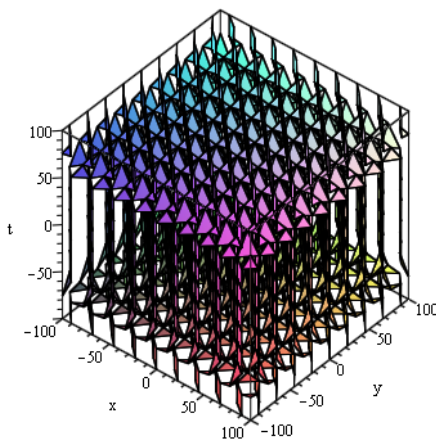


Figure 12: For real part $-100 \leq x \leq 100, -100 \leq y \leq 100, -100 \leq t \leq 100$

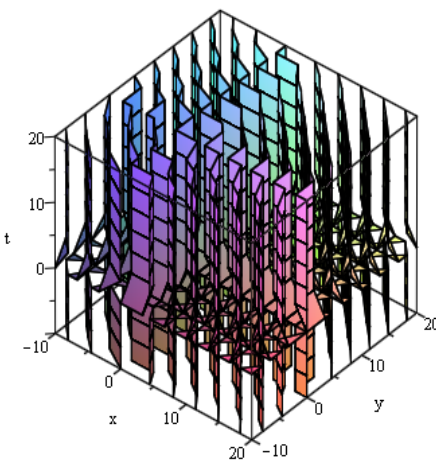


Figure 13: For imaginary part $-10 \leq x \leq 20, -10 \leq y \leq 20, -10 \leq t \leq 20$

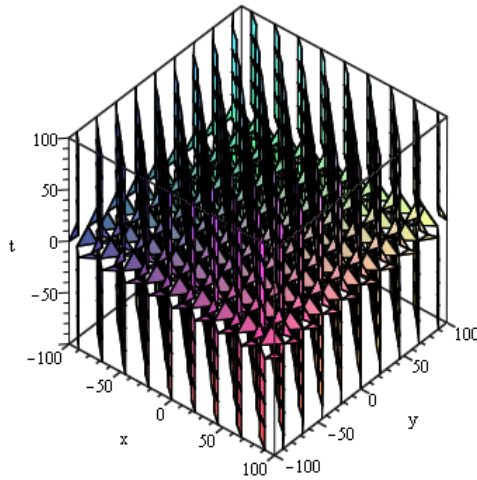


Figure 14: For imaginary part $-100 \leq x \leq 100, -100 \leq y \leq 100, -100 \leq t \leq 100$

Example 4: Consider the following three dimensional Schrödinger equation

$$i \frac{\partial \psi(x,y,z,t)}{\partial t} = -\frac{1}{2} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) + V(x,y,z)\psi + |\psi|^2 \psi, \quad (19)$$

$$(x, y, z) \in [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi],$$

subject to the initial condition

$$\psi(x, y, z, 0) = \sin x \sin y \sin z,$$

$$\text{where } V(x, y, z) = 1 - \sin^2 x \sin^2 y \sin^2 z,$$

To solve equation (19) by HAM, the linear operator is defined as

$$L[\varphi(x, y, z, t; q)] = \frac{\partial^\alpha}{\partial t^\alpha} [\varphi(x, y, z, t; q)], \quad 0 < \alpha \leq 1 \quad L^{-1} = J_*^\alpha(.)$$

with the property $L[C] = 0$, where C is constant of integration and the non-linear operator is defined as

$$N[\varphi(x, y, z, t; q)] = -\frac{\partial}{\partial t} \varphi(x, y, z, t; q) + \frac{i}{2} \left(\frac{\partial^2}{\partial x^2} \varphi(x, y, z, t; q) + \frac{\partial^2}{\partial y^2} \varphi(x, y, z, t; q) + \frac{\partial^2}{\partial z^2} \varphi(x, y, z, t; q) \right) - i\varphi(x, y, z, t; q)V(x, y, z) - i|\varphi(x, y, z, t; q)|^2 \varphi(x, y, z, t; q).$$

The zeroth-order deformation is,

$$(1 - q)L[\varphi(x, y, z, t; q) - \psi_0(x, y, z, t)] = qhN[\varphi(x, y, z, t; q)]. \quad (20)$$

where $q \in [0, 1]$ is an embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter; $\psi_0(x, y, z, t)$ is initial guess.

$$\text{for } q = 0, \quad \varphi(x, y, z, t; 0) = \psi_0(x, y, z, t),$$

$$\text{for } q = 1, \quad \varphi(x, y, z, t; 1) = \psi(x, y, z, t),$$

thus as q increases from 0 to 1 the solution $\varphi(x, y, z, t; q)$ varies from the initial guess $\psi_0(x, y, z, t)$ to the solution $\psi(x, y, z, t)$. By Taylor's series expansion, we have

$$\begin{aligned} \varphi(x, y, z, t; q) &= \\ &\varphi(x, y, z, t; 0) + q \frac{\partial}{\partial q} \varphi(x, y, z, t; 0) + q^2 \frac{\partial^2}{\partial q^2} \varphi(x, y, z, t; 0) + \dots + q^3 \frac{\partial^3}{\partial q^3} \varphi(x, y, z, t; 0) + \dots, \\ &= \psi_0(x, y, z, t) + \sum_{m=1}^{\infty} \psi_m q^m, \end{aligned} \quad (21)$$

$$\psi_m = \frac{1}{m!} \frac{\partial^m \varphi(x, y, z, t; q)}{\partial q^m}, \quad \text{at } q = 0,$$

For $q = 1$ eq (3) implies

$$\psi(x, y, z, t) = \psi_0(x, y, z, t) + \sum_{m=1}^{\infty} \psi_m(x, y, z, t).$$

Differentiating (2) wrt embedding parameter q , setting $q = 0$ and dividing by $m!$, the m th-order deformation is

$$L[\psi_m(x, y, z, t) - \chi_m \psi_{m-1}(x, y, z, t)] = h R_m(\psi_{m-1}),$$

$$\text{where } R_m(\psi_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\varphi(x, y, z, t; q)]}{\partial q^{m-1}}.$$

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases}$$

By taking $\psi_0(x, y, z, t) = \sin x \sin y \sin z$, differentiating (2) w.r.t q and setting $q = 0$, we have

$$\begin{aligned} L\left[\frac{\partial}{\partial q} \varphi(x, y, z, t; 0)\right] &= hN[\varphi(x, y, z, t; 0)], \\ L[\psi_1(x, y, z, t)] &= h \left[-\frac{\partial}{\partial t} \varphi(x, y, z, t) + \frac{i}{2} \left(\frac{\partial^2}{\partial x^2} \varphi(x, y, z, t; 0) + \frac{\partial^2}{\partial y^2} \varphi(x, y, z, t; 0) + \frac{\partial^2}{\partial z^2} \varphi(x, y, z, t; 0) \right) - \right. \\ &\quad \left. i\varphi(x, y, t; 0) - \sin 2x \sin 2y \sin 2z - i\varphi(x, y, z, t; 0) \right], \end{aligned}$$

$$L[\psi_1(x, y, z, t)] = h \left[-\frac{\partial}{\partial t} \sin x \sin y \sin z + \frac{i}{2} \left(\frac{\partial^2}{\partial x^2} \sin x \sin y \sin z + \frac{\partial^2}{\partial y^2} \sin x \sin y \sin z + \frac{\partial^2}{\partial z^2} \sin x \sin y \sin z \right) - i \sin x \sin y \sin z \right]$$

$$\psi_1(x, y, z, t) = hL^{-1} \left[-\frac{5}{2} i \sin x \sin y \sin z \right],$$

$$= h J^\alpha \left[-\frac{5}{2} i \sin x \sin y \sin z \right],$$

$$= h \left[-\frac{5}{2} i \sin x \sin y \sin z \frac{t^\alpha}{\Gamma(\alpha+1)} \right],$$

$$\psi_2(x, y, z, t) = h^2 \left[-\frac{25}{4} i \sin x \sin y \sin z \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right],$$

$$\psi_3(x, y, z, t) = h^3 \left[\frac{125}{8} i \sin x \sin y \sin z \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} \right],$$

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The series solution is given by

$$\psi(x, y, z, t) = \psi_0(x, y, z, t) + \psi_1(x, y, z, t) + \psi_2(x, y, z, t) + \psi_3(x, y, z, t) + \dots,$$

$$\psi(x, y, z, t) = \sin x \sin y \sin z \left[1 + h \left[-\frac{5}{2} i \frac{t^\alpha}{\Gamma(\alpha+1)} \right] + h^2 \left[-\frac{25}{4} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right] + h^3 \left[\frac{125}{8} i \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \right] \right] + \dots,$$

For $\alpha = 1$ and $h = 1$, we have

$$\psi(x, y, z, t) = \sin x \sin y \sin z \left[1 + \left(-\frac{5}{2} it\right) + \frac{1}{2!} \left(-\frac{5}{2} it\right)^2 + \frac{1}{3!} \left(-\frac{5}{2} it\right)^3 + \dots \right],$$

$$= \sin x \sin y \sin z e^{-\frac{5}{2} it},$$

which is the exact solution.

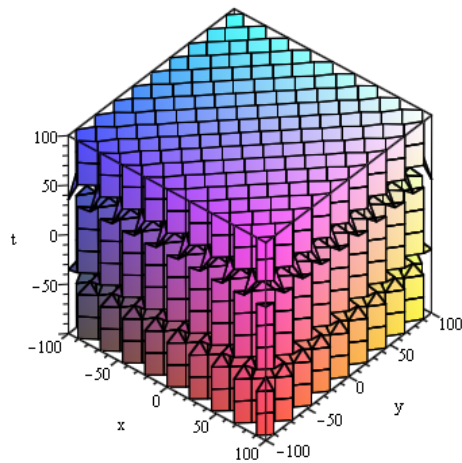


Figure 15: For real part $\sin(z) = -1$, $-100 \leq x \leq 100$, $-100 \leq y \leq 100$, $-100 \leq t \leq 100$,

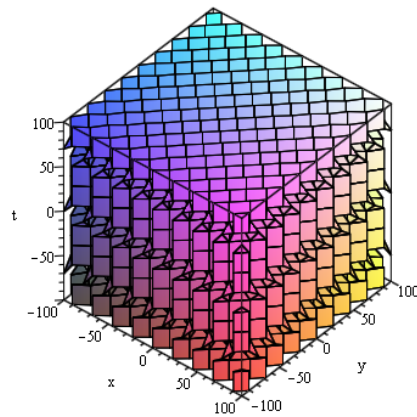


Figure 16: For imaginary part $\sin(z) = -1$, $-100 \leq x \leq 100$, $-100 \leq y \leq 100$, $-100 \leq t \leq 100$,

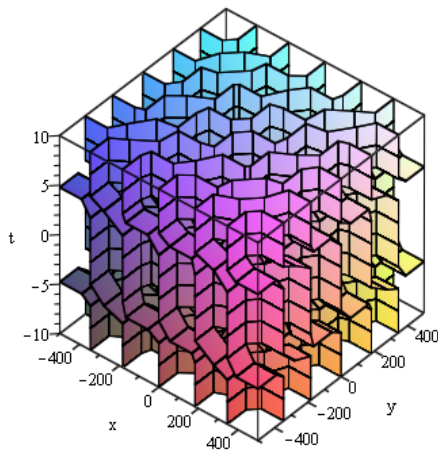


Figure 17: For real part $\sin(z) = -1$, $-500 \leq x \leq 500$, $-500 \leq y \leq 500$, $-10 \leq t \leq 10$,

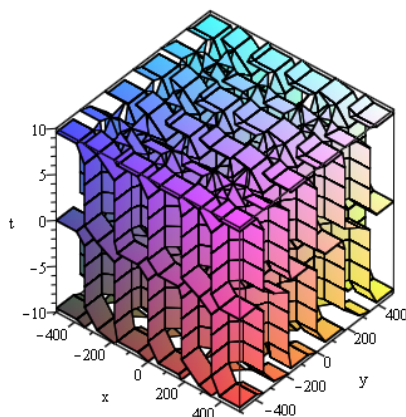


Figure 18: For imaginary part $\sin(z) = -1$, $-500 \leq x \leq 500$, $-500 \leq y \leq 500$, $-10 \leq t \leq 10$.

4. Conclusions

Homotopy Analysis Method (HAM) is implemented to obtain analytical solutions of nonlinear Schrödinger equations. Numerical results and graphical representations clearly reflect complete compatibility of the proposed algorithm and discussed problems.

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