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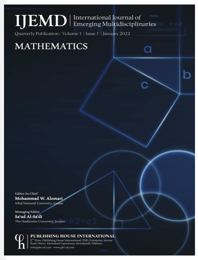

A New Companion Inequality of Ostrowski's type with Applications to P.D.F.'s

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Abstract

In this paper, variant inequalities of Ostrowski's type for absolutely continuous mappings whose derivatives are monotonic, belongs to $L_1[a,b]$, $L_q[a,b]$, $(q>1)$ and $L_\infty[a,b]$ are established.

Keywords: *Ostrowski inequality, Bounded variation, Absolutely continuous*
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1. Introduction

In 1938, Ostrowski established a very interesting inequality for differentiable mappings with bounded derivatives, as follows [6]:

Theorem 1.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'(x)| \leq M$, then the following inequality,*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{1.1}$$

holds for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

For recent results concerning Ostrowski's inequality see [1, 2]. Also, the reader may be refer to the monograph [6] where various inequalities of Ostrowski type are discussed.

In [5], Guessab and Schmeisser have proved among others, the following companion of Ostrowski's inequality:

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be satisfies the Lipschitz condition, i.e., $|f(t) - f(s)| \leq M|t-s|$. Then for each $x \in [a, \frac{a+b}{2}]$, we have the inequality*

$$\left| \frac{f(x) + f(a+b-x)}{2} - \int_a^b f(t)\,dt \right| \leq \left[\frac{1}{8} + 2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)M \tag{1.2}$$

for any $x \in [a, \frac{a+b}{2}]$. The inequality $1/8$ is best possible in (1.2) in the sense that it cannot be replaced by a smaller constant.

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We may also note that the best inequality in (1.2) is obtained for $x = \frac{3a+b}{4}$, giving the trapezoid type inequality

$$\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \frac{(b-a)}{8} M \tag{1.3}$$

The constant $1/8$ is sharp in (1.3) in the sense mentioned above.

Companions of Ostrowski's integral inequality for absolutely continuous functions was considered by Dragomir in [7], as follows :

Theorem 1.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then we have*

$$\left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t)\, dt \right|$$
$$\le \begin{cases} \left[\frac{1}{8} + 2\left(\frac{x - \frac{3a+b}{4}}{b-a} \right)^2 \right] (b-a) \|f'\|_\infty, \quad f' \in L_\infty[a,b] \\ \frac{2^{1/q}}{(q+1)^{q+1}} \left[\left(\frac{x-a}{b-a} \right)^{q+1} - \left(\frac{\frac{a+b}{2} - a}{b-a} \right)^{q+1} \right]^{1/q} (b-a)^{1/q} \|f'\|_{[a,b],p}, \\ \qquad p > 1, \frac{1}{p} + \frac{1}{q} = 1, \text{ and } f' \in L_p[a,b] \\ \left[\frac{1}{4} + \left| \frac{x - \frac{3a+b}{4}}{b-a} \right| \right] \|f'\|_{[a,b],1} \end{cases} \tag{1.4}$$

for all $x \in [a, \frac{a+b}{2}]$.

In [8], S.S. Dragomir established some inequalities for the this companion for mappings of bounded variation. Also, Z. Liu [9], introduced some companions of an Ostrowski type integral inequality for functions whose derivatives are absolutely continuous. Recently, N.S. Barnett et al. [4], have proved some companions for the Ostrowski inequality and for the generalized trapezoid inequality. For more related works the reader may refer to [1]–[10].

The aim of this paper is to study the companion of Ostrowski inequality (1.2) for the class of functions whose derivatives in absolutely continuous.

2. Inequalities for Mappings of Bounded Variation

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then for all $x \in [a, \frac{a+b}{2})$, we have the inequality*

$$\left| \left(\frac{a+b}{2} - x \right) (f(x) + f(a+b-x)) - \int_x^{a+b-x} f(t)\, dt \right| \le \left(\frac{a+b}{2} - x \right) \cdot \bigvee_x^{a+b-x} (f), \tag{2.1}$$

where $\bigvee_x^{a+b-x}(f)$ denotes to total variation of f over $[x, a+b-x]$, $x \in [a, \frac{a+b}{2})$.

Proof. Using the integration by parts formula for Riemann–Stieltjes integral, we have

$$\int_x^{a+b-x} \left(t - \frac{a+b}{2} \right) df(t) = \left(\frac{a+b}{2} - x \right) (f(x) + f(a+b-x)) - \int_x^{a+b-x} f(t)\, dt$$

for all $x \in [a, \frac{a+b}{2}]$.

Now, We use the fact that for a continuous function $p : [c,d] \to \mathbb{R}$ and a function $\nu : [c,d] \to \mathbb{R}$ of bounded variation, one has the inequality

$$\left| \int_c^d p(t)\, d\nu(t) \right| \le \sup_{t \in [c,d]} |p(t)| \bigvee_a^b (\nu). \tag{2.2}$$

Applying the inequality (2.2) for $p(t) = t - \frac{a+b}{2}$, as above and $\nu(t) = f(t)$, $t \in [a,b]$, we get

$$\left| \int_x^{a+b-x} \left(t - \frac{a+b}{2} \right) df(t) \right| \le \sup_{t \in [x, a+b-x]} \left| \left(t - \frac{a+b}{2} \right) \right| \cdot \bigvee_x^{a+b-x} (f) \le \left(\frac{a+b}{2} - x \right) \cdot \bigvee_x^{a+b-x} (f),$$

for all $x \in [a, \frac{a+b}{2}]$, which completes the proof.

□

Corollary 2.2. *In Theorem 2.1. Choose $x = a$, then we get*

$$\left| (b-a) \frac{f(a) + f(b)}{2} - \int_a^b f(t)\, dt \right| \le \frac{(b-a)}{2} \cdot \bigvee_a^b (f). \tag{2.3}$$

Corollary 2.3. *Let f as in Theorem 2.1. Additionally, if f is symmetric about the x-axis, i.e., $f(a+b-x) = f(x)$, we have*

$$\left| (a+b-2x) f(x) - \int_x^{a+b-x} f(t)\, dt \right| \le \left(\frac{a+b}{2} - x \right) \cdot \bigvee_x^{a+b-x} (f), \tag{2.4}$$

for all $x \in [a, \frac{a+b}{2})$.

Corollary 2.4. *In Corollary 2.3. Choose $x = a$, then we get*

$$\left| (b-a) f(a) - \int_a^b f(t)\, dt \right| \le \frac{(b-a)}{2} \cdot \bigvee_a^b (f). \tag{2.5}$$

Therefore, we may write the following result regarding monotonic mappings:

Corollary 2.5. *Let $f : [a,b] \to \mathbb{R}$ be a monotonous mapping on $[a,b]$. Then for all $x \in [a, \frac{a+b}{2})$, we have the inequality*

$$\left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{a+b-2x} \int_x^{a+b-x} f(t)\, dt \right| \le \left(\frac{a+b}{2} - x \right) \cdot |f(b) - f(a)|. \tag{2.6}$$

The following result holds for L–lipschitz mappings:

Corollary 2.6. *Let $f : [a,b] \to \mathbb{R}$ be a L–lipschitz mapping on $[a,b]$. Then for all $x \in [a, \frac{a+b}{2})$, we have the inequality*

$$\left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{a+b-2x} \int_x^{a+b-x} f(t)\, dt \right| \le L \left(\frac{a+b}{2} - x \right) (b-a). \tag{2.7}$$

Remark 2.7. *If we assume that f is continuous differentiable on (a,b) and f' is integrable on (a,b), then we have*

$$\left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{a+b-2x} \int_x^{a+b-x} f(t)\, dt \right| \le \left(\frac{a+b}{2} - x \right) \|f'\|_1, \tag{2.8}$$

for all $x \in [a, \frac{a+b}{2})$.

3. Inequalities for Mappings whose First Derivatives Belong to $L_p[a,b]$, $(1 < p \le \infty)$

We may state the following theorem.

Theorem 3.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I°, the interior of the interval I, where $a, b \in I$ with $a < b$, such that $f' \in L_1[a,b]$. If f' is bounded on $[a,b]$, i.e., $\|f'\|_\infty := \sup_{t \in [a,b]} |f'(t)| < \infty$, then we have the following inequality:*

$$\left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{a+b-2x} \int_x^{a+b-x} f(t)\, dt \right| \le \frac{(a+b-2x)}{4} \|f'\|_\infty, \tag{3.1}$$

for all $x \in [a, \frac{a+b}{2})$.

Proof. Integrating by parts

$$\frac{f(x)+f(a+b-x)}{2}-\frac{1}{a+b-2x}\int_x^{a+b-x}f(t)\,dt=\frac{1}{a+b-2x}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)f'(t)\,dt \tag{3.2}$$

for all $x\in[a,\frac{a+b}{2})$.

Therefore, since f' is bounded on $[a,b]$, we have

$$\begin{aligned}&\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{a+b-2x}\int_x^{a+b-x}f(t)\,dt\right|\\&=\frac{1}{a+b-2x}\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'(t)\right|dt\\&\le\frac{\|f'\|_\infty}{a+b-2x}\left[\int_x^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)dt+\int_{\frac{a+b}{2}}^{a+b-x}\left(t-\frac{a+b}{2}\right)dt\right]\\&=\frac{(a+b-2x)}{4}\|f'\|_\infty,\end{aligned}$$

for all $x\in[a,\frac{a+b}{2})$, which completes the proof. □

Corollary 3.2. *In Theorem 3.1. Choose $x=a$, then we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{(b-a)}{4}\|f'\|_\infty. \tag{3.3}$$

Corollary 3.3. *Let f as in Theorem 3.1. Additionally, if f is symmetric about the x-axis, i.e., $f(a+b-x)=f(x)$, we have*

$$\left|f(x)-\frac{1}{a+b-2x}\int_x^{a+b-x}f(t)\,dt\right|\le\frac{(a+b-2x)}{4}\|f'\|_\infty, \tag{3.4}$$

for all $x\in[a,\frac{a+b}{2})$.

Corollary 3.4. *In Corollary 3.3. Choose $x=a$, then we get*

$$\left|f(a)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{(b-a)}{4}\|f'\|_\infty. \tag{3.5}$$

We may state the following theorem.

Theorem 3.5. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be an absolutely continuous mapping on I°, the interior of the interval I, where $a,b\in I$ with $a<b$, such that $f'\in L_1[a,b]$. If f' is belong to $L_p[a,b]$, $p>1$, then we have the following inequality:*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{a+b-2x}\int_x^{a+b-x}f(t)\,dt\right|\le\frac{(a+b-2x)^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\|f'\|_{p,[x,a+b-x]} \tag{3.6}$$

for all $x\in[a,\frac{a+b}{2})$.

Proof. Using Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{a+b-2x}\int_x^{a+b-x}f(t)\,dt\right| \\
&=\frac{1}{a+b-2x}\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'(t)\right|dt \\
&\leq\frac{1}{a+b-2x}\left(\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|^q dt\right)^{1/q}\left(\int_x^{a+b-x}\left|f'(t)\right|^p dt\right)^{1/p} \\
&=\frac{\|f'\|_{p,[x,a+b-x]}}{a+b-2x}\left(\int_x^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)^q dt+\int_{\frac{a+b}{2}}^{a+b-x}\left(t-\frac{a+b}{2}\right)^q dt\right)^{1/q} \\
&=\frac{\|f'\|_{p,[x,a+b-x]}}{a+b-2x}\left(\frac{2}{q+1}\right)^{1/q}\left(\frac{a+b}{2}-x\right)^{(q+1)/q} \\
&=\frac{(a+b-2x)^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\left\|f'\right\|_{p,[x,a+b-x]},
\end{aligned}$$

for all $x\in[a,\frac{a+b}{2})$, which completes the proof. □

Corollary 3.6. *In Theorem 3.5. Choose $x=a$, then we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{(b-a)^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\left\|f'\right\|_{p,[a,b]}. \tag{3.7}$$

Corollary 3.7. *Let f as in Theorem 3.5. Additionally, if f is symmetric about the x-axis, i.e., $f(a+b-x)=f(x)$, we have*

$$\left|(a+b-2x)f(x)-\int_x^{a+b-x}f(t)\,dt\right|\leq\frac{(a+b-2x)^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\left\|f'\right\|_{p,[x,a+b-x]}, \tag{3.8}$$

for all $x\in[a,\frac{a+b}{2})$.

In Theorem 3.5. Choose $x=a$, then we get

$$\left|f(a)-\int_a^b f(t)\,dt\right|\leq\frac{(b-a)^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\left\|f'\right\|_{p,[a,b]}. \tag{3.9}$$

4. Inequalities for Convex Mappings

Theorem 4.1. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be an absolutely continuous mapping on I°, the interior of the interval I, where $a,b\in I$ with $a<b$, such that $f'\in L_1[a,b]$. If $|f'|$ is convex on $[a,b]$, then we have the following inequality:*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{a+b-2x}\int_x^{a+b-x}f(t)\,dt\right|\leq\frac{(a+b-2x)}{8}\left(\left|f'(x)\right|+\left|f'(a+b-x)\right|\right), \tag{4.1}$$

for all $x\in[a,\frac{a+b}{2})$.

Proof. Since $|f'|$ is convex on $[a,b]$ and therefore, on $[x,a+b-x]$, we have

$$\left|f'(t)\right|\leq\frac{t-x}{a+b-2x}\cdot\left|f'(a+b-x)\right|+\frac{a+b-x-t}{a+b-2x}\cdot\left|f'(x)\right|,\quad\forall t\in(x,a+b-x];$$

Thus,

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{a+b-2x}\int_x^{a+b-x}f(t)\,dt\right|$$
$$=\frac{1}{a+b-2x}\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'(t)\right|dt$$
$$\leq\frac{1}{a+b-2x}\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left[\frac{t-x}{a+b-2x}\cdot\left|f'(a+b-x)\right|+\frac{a+b-x-t}{a+b-2x}\cdot\left|f'(x)\right|\right]dt$$
$$=\frac{(a+b-2x)}{8}\left(\left|f'(x)\right|+\left|f'(a+b-x)\right|\right),$$

for all $x\in[a,\frac{a+b}{2})$. □

Corollary 4.2. *In Theorem 4.1. Choose $x=a$, then we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{(b-a)}{8}\cdot\left(\left|f'(a)\right|+\left|f'(b)\right|\right). \tag{4.2}$$

Corollary 4.3. *Let f as in Theorem 4.1. Additionally, if f is symmetric about the x-axis, i.e., $f(a+b-x)=f(x)$, we have*

$$\left|f(x)-\frac{1}{a+b-2x}\int_x^{a+b-x}f(t)\,dt\right|\leq\frac{(a+b-2x)}{8}\left(\left|f'(x)\right|+\left|f'(a+b-x)\right|\right), \tag{4.3}$$

for all $x\in[a,\frac{a+b}{2})$.

Corollary 4.4. *In Corollary 4.3. Choose $x=a$, then we get*

$$\left|f(a)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{(b-a)}{8}\left(\left|f'(a)\right|+\left|f'(b)\right|\right). \tag{4.4}$$

Theorem 4.5. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be an absolutely continuous mapping on I°, the interior of the interval I, where $a,b\in I$ with $a<b$, such that $f'\in L_1[a,b]$. If $|f'|^q$, $q>1$ is convex on $[a,b]$, then we have the following inequality:*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{a+b-2x}\int_x^{a+b-x}f(t)\,dt\right|\leq\frac{(a+b-2x)}{2^{1+\frac{1}{q}}(p+1)^{1/p}}\left(\left|f'(x)\right|^q+\left|f'(a+b-x)\right|^q\right)^{1/q}, \tag{4.5}$$

for all $x\in[a,\frac{a+b}{2})$, where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$.

Proof. Since $|f'|^q$ is convex and using the Hölder inequality, we have

$$\left|(f(x)+f(a+b-x))-\frac{1}{a+b-2x}\int_x^{a+b-x}f(t)\,dt\right|$$
$$=\frac{1}{a+b-2x}\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'(t)\right|dt$$
$$\leq\frac{1}{a+b-2x}\left(\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|^q dt\right)^{1/q}\left(\int_x^{a+b-x}\left|f'(t)\right|^p dt\right)^{1/p}$$

Since $|f'|^q$ is convex on $[a,b]$ and therefore on $[x,a+b-x]$, we have

$$\left|f'(t)\right|^q\leq\frac{t-x}{a+b-2x}\cdot\left|f'(a+b-x)\right|^q+\frac{a+b-x-t}{a+b-2x}\cdot\left|f'(x)\right|^q,\quad\forall t\in[x,a+b-x];$$

which follows that,

$$
\begin{aligned}
&\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{a+b-2x}\int_x^{a+b-x} f(t)\,dt\right| \\
&\leq \frac{1}{a+b-2x}\left(\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|^p dt\right)^{1/p} \\
&\qquad \times\left(\int_x^{a+b-x}\left[\frac{t-x}{a+b-2x}\cdot\left|f'(a+b-x)\right|^q+\frac{a+b-x-t}{a+b-2x}\cdot\left|f'(x)\right|^q\right]dt\right)^{1/q}. \\
&=\frac{1}{a+b-2x}\left(\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|^p dt\right)^{1/p}\left(\frac{a+b}{2}-x\right)^{1/q}\left(\left|f'(x)\right|^q+\left|f'(a+b-x)\right|^q\right)^{1/q} \\
&=\frac{1}{a+b-2x}\left(\frac{2}{(p+1)}\left(\frac{a+b}{2}-x\right)^{p+1}\right)^{1/p}\left(\frac{a+b}{2}-x\right)^{1/q}\left(\left|f'(x)\right|^q+\left|f'(a+b-x)\right|^q\right)^{1/q} \\
&=\frac{1}{a+b-2x}\frac{1}{(p+1)^{1/p}}\frac{2^{\frac{1}{p}}}{2^{1+\frac{1}{p}+\frac{1}{q}}}(a+b-2x)^{1+\frac{1}{p}+\frac{1}{q}}\left(\left|f'(x)\right|^q+\left|f'(a+b-x)\right|^q\right)^{1/q} \\
&=\frac{(a+b-2x)}{2^{1+\frac{1}{q}}(p+1)^{1/p}}\left(\left|f'(x)\right|^q+\left|f'(a+b-x)\right|^q\right)^{1/q},
\end{aligned}
$$

since $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, which completes the proof. □

Corollary 4.6. *In Theorem 4.5. Choose $x=a$, then we get*

$$
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)}{2^{1+\frac{1}{q}}(p+1)^{1/p}}\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{1/q}. \tag{4.6}
$$

Corollary 4.7. *Let f as in Theorem 4.5. Additionally, if f is symmetric about the x-axis, i.e., $f(a+b-x)=f(x)$, we have*

$$
\left|f(x)-\frac{1}{a+b-2x}\int_x^{a+b-x} f(t)\,dt\right| \leq \frac{(a+b-2x)}{2^{1+\frac{1}{q}}(p+1)^{1/p}}\left(\left|f'(x)\right|^q+\left|f'(a+b-x)\right|^q\right)^{1/q}, \tag{4.7}
$$

for all $x\in[a,\frac{a+b}{2})$.

Corollary 4.8. *In Corollary 4.7. Choose $x=a$, then we get*

$$
\left|f(a)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)}{2^{1+\frac{1}{q}}(p+1)^{1/p}}\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{1/q}. \tag{4.8}
$$

Theorem 4.9. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be an absolutely continuous mapping on I°, the interior of the interval I, where $a,b\in I$ with $a<b$, such that $f'\in L_1[a,b]$. If $|f'|^q$, $q>1$ is concave on $[a,b]$, then we have the following inequality:*

$$
\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{a+b-2x}\int_x^{a+b-x} f(t)\,dt\right| \leq \frac{(a+b-2x)}{2(1+p)^{1/p}}\left|f'\left(\frac{a+b}{2}\right)\right|, \tag{4.9}
$$

for all $x\in[a,\frac{a+b}{2})$, where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$.

Proof. Using Hölder inequality, we have

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{a+b-2x}\int_x^{a+b-x}f(t)\,dt\right|$$
$$=\frac{1}{a+b-2x}\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'(t)\right|dt$$
$$\leq\frac{1}{a+b-2x}\left(\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|^p dt\right)^{1/p}\left(\int_x^{a+b-x}\left|f'(t)\right|^q dt\right)^{1/q}.$$

Now, let us write,

$$\int_x^{a+b-x}\left|f'(t)\right|^q dt=(a+b-2x)\int_0^1\left|f'(\lambda(a+b-x)+(1-\lambda)x)\right|^q d\lambda.$$

Since $|f|^q$, $q>1$ is concave on $[a,b]$ and therefore on $[x,a+b-x]$, we can use the integral Jensen's inequality to obtain

$$(a+b-2x)\int_0^1\left|f'(\lambda(a+b-x)+(1-\lambda)x)\right|^q d\lambda$$
$$=(a+b-2x)\int_0^1\lambda^0\left|f'(\lambda(a+b-x)+(1-\lambda)x)\right|^q d\lambda$$
$$\leq(a+b-2x)\left(\int_0^1\lambda^0 d\lambda\right)\left|f'\left(\frac{1}{\int_0^1\lambda^0 d\lambda}\int_0^1(\lambda(a+b-x)+(1-\lambda)x)\,d\lambda\right)\right|^q$$
$$=(a+b-2x)\left|f'\left(\frac{a+b}{2}\right)\right|^q.$$

Combining all obtained inequalities, we get

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{a+b-2x}\int_x^{a+b-x}f(t)\,dt\right|$$
$$\leq\frac{1}{a+b-2x}\left(\frac{2}{(p+1)}\left(\frac{a+b}{2}-x\right)^{p+1}\right)^{1/p}(a+b-2x)^{1/q}\left|f'\left(\frac{a+b}{2}\right)\right|$$
$$=\frac{(a+b-2x)}{2(1+p)^{1/p}}\left|f'\left(\frac{a+b}{2}\right)\right|,$$

for all $x\in[a,\frac{a+b}{2})$, where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, which is required. □

Corollary 4.10. *In Theorem 4.9. Choose $x=a$, then we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{(b-a)}{2(1+p)^{1/p}}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{4.10}$$

Corollary 4.11. *Let f as in Theorem 4.9. Additionally, if f is symmetric about the x-axis, i.e., $f(a+b-x)=f(x)$, we have*

$$\left|f(x)-\frac{1}{a+b-2x}\int_x^{a+b-x}f(t)\,dt\right|\leq\frac{(a+b-2x)^2}{2(1+p)^{1/p}}\left|f'\left(\frac{a+b}{2}\right)\right|, \tag{4.11}$$

for all $x\in[a,\frac{a+b}{2})$, where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, which is required.

Corollary 4.12. *In Corollary 4.11. Choose $x=a$, then we get*

$$\left|(b-a)f(a)-\int_a^b f(t)\,dt\right|\leq\frac{(b-a)}{2(1+p)^{1/p}}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{4.12}$$

5. Applications for P.D.F.'s

Let X be a random variable taking values in the finite interval $[a,b]$, with the probability density function $f:[a,b]\to[0,1]$ with the cumulative distribution function $F(x)=Pr(X\le x)=\int_a^b f(t)dt$.

Theorem 5.1. *With the assumptions of Theorem 2.1, we have the inequality*

$$\left|\frac{1}{2}\left[F(x)+F(a+b-x)\right]-\frac{b-E(X)}{b-a}\right| \le \frac{(x-a)^2}{6(b-a)}\left(\left|F'(a)\right|+\left|F'(b)\right|\right)+\frac{8(x-a)^2+3(a+b-2x)^2}{24(b-a)}\left(\left|F'(x)\right|+\left|F'(a+b-x)\right|\right)$$

for all $x\in[a,\frac{a+b}{2}]$, where $E(X)$ is the expectation of X.

Proof. In the proof of Theorem 2.1, let $f=F$, and taking into account that

$$E(X)=\int_a^b t dF(t)=b-\int_a^b F(t)dt.$$

We left the details to the interested reader. □

Corollary 5.2. *In Theorem 5.1, choose $x=\frac{3a+b}{4}$, we get*

$$\left|\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right]-\frac{b-E(X)}{b-a}\right| \le \frac{(b-a)}{96}\left[\left|F'(a)\right|+5\left|F'\left(\frac{3a+b}{4}\right)\right|+5\left|F'\left(\frac{a+3b}{4}\right)\right|+\left|F'(b)\right|\right].$$

Corollary 5.3. *In Theorem 5.1, if F is symmetric about the x-axis, i.e., $F(a+b-x)=F(x)$, we have*

$$\left|F(x)-\frac{b-E(X)}{b-a}\right| \le \frac{(x-a)^2}{6(b-a)}\left(\left|F'(a)\right|+\left|F'(b)\right|\right) + \frac{8(x-a)^2+3(a+b-2x)^2}{24(b-a)}\left(\left|F'(x)\right|+\left|F'(a+b-x)\right|\right)$$

for all $x\in[a,\frac{a+b}{2}]$.

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Competing Interests

The author declare no competing interests.

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