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Inequalities for Riemann–Stieltjes integral

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Abstract

Two new inequalities for Riemann–Stieltjes integral are introduced for functions of bounded p-variation and Hölder continuous integrators.

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1. Introduction

If $[a,b]$ is a compact interval , a set of points $P := \{x_0, x_1, \cdots, x_n\}$, satisfying the inequalities

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b,$$

is called a partition of $[a,b]$. The interval $[x_{i-1}, x_i]$ is called i-th subinterval of P and we write $\Delta x_i = x_i - x_{i-1}$, so that $\sum_{i=1}^{n} \Delta x_i = b-a$. The collection of all possible partitions of $[a,b]$ will be denoted by $\mathscr{P}[a,b]$.

Definition 1.1. *[3] Let f be defined on $[a,b]$. If $P := \{x_0, x_1, \cdots, x_n\}$ is a partition of $[a,b]$, write*

$$\Delta f_i = f(x_i) - f(x_{i-1}),$$

for $i = 1, 2, \cdots, n$. If there exists a positive number M such that $\nu_p(f) := \left(\sum_{i=1}^{n} |\Delta f_i|^p\right)^{\frac{1}{p}} \le M$, $(1 \le p < \infty)$ for all partition of $[a,b]$, then f is said to be of bounded p–variation $\nu_p(f)$ on $[a,b]$.

where ν_1 is the ordinary class of functions of bounded variation and there is strict inclusion. Consequently, Jensen's inequality implies that $\nu_p(f) \subset \nu_q(f)$, for $1 \le p < q < \infty$, i.e., the class of $\nu_p(f)$ is a proper subset of $\nu_q(f)$ whenever $1 \le p < q < \infty$.

Let f be of bounded variation on $[a,b]$, and let $\sum(P)$ denote the sum $\left(\sum_{i=1}^{n} |\Delta f_i|^p\right)^{\frac{1}{p}}$ corresponding to the partition P of $[a,b]$. The number

$$\bigvee_a^b (f;p) = \sup\left\{\sum(P) : P \in \mathscr{P}[a,b]\right\}, \quad 1 \le p < \infty$$

is called the total p–variation of f on the interval $[a,b]$.

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We recall that a function $f : I \to \mathbb{R}$ is said to satisfy a Lipschitz condition of order α, $\alpha > 0$ if there exists a positive number L such that

$$|f(x) - f(c)| < L|x-c|^{\alpha}. \tag{1.1}$$

Moreover, if $0 < \alpha \le 1$, then f is said to satisfy a Hölder condition.
In 1924, Wiener [5], showed that $\mathrm{Lip}_{\frac{1}{p}}(f) \subset \nu_p(f)$, where $\mathrm{Lip}_{\frac{1}{p}}(f)$ is the class of functions satisfying the Lipschitz condition of order p. More preciously, if f has the α-Hölder property, then f has bounded p-variation with $p = \frac{1}{\alpha}$. A continuous function of bounded p-variation for some $1 \le p < \infty$ need not have the α-Holder property. As pointed out in [4], the series

$$\sum_{k=1}^{\infty} \frac{\sin kt}{k \log k}, \quad 0 \le t \le 1,$$

converges uniformly to the sum g, which is absolutely continuous and, hence, has bounded p-variation for each $1 \le p < \infty$. However, this g satisfies no Holder property of order $\alpha > 0$, for more details the reader may refer to [2], Section 10.6.1 (2).

Theorem 1.2. *([3]) Let $p, \nu : [a,b] \to \mathbb{R}$ such that p is a continuous function on $[a,b]$ and ν of bounded variation, then $\int_a^b p(t)\, d\nu(t)$ exits and the inequality*

$$\left| \int_a^b p(t)\, d\nu(t) \right| \le \sup_{t \in [a,b]} |p(t)| \bigvee_a^b (\nu) \tag{1.2}$$

holds.

Theorem 1.3. *([4]) If $\bigvee_a^b (f;p)$ and $\bigvee_a^b (g;q) < \infty$ with $\frac{1}{p} + \frac{1}{q} > 1$, and f and g have no common discontinuities, then the Riemann–Stieltjes integral $\int_a^b f(t)\, dg(t)$ exists and*

$$\left| \int_a^b f(t)\, dg(t) \right| \le \left[1 + \zeta\left(\frac{1}{p} + \frac{1}{q} \right) \right] \cdot \left[|f(a)| + \bigvee_a^b (f;p) \right] \cdot \bigvee_a^b (g;q), \tag{1.3}$$

where, $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function.

Therefore, we may deduce the following result:

Corollary 1.4. *Let $f, g : [a,b] \to \mathbb{R}$ be two functions of bounded variation on $[a,b]$, then the Riemann–Stieltjes integral $\int_a^b f(t)\, dg(t)$ exists and*

$$\left| \int_a^b f(t)\, dg(t) \right| \le \left[1 + \frac{\pi^2}{6} \right] \cdot \left[|f(a)| + \bigvee_a^b (f) \right] \cdot \bigvee_a^b (g), \tag{1.4}$$

In special case, if $f(a) = 0$, then

$$\left| \int_a^b f(t)\, dg(t) \right| \le \left[1 + \frac{\pi^2}{6} \right] \cdot \bigvee_a^b (f) \cdot \bigvee_a^b (g). \tag{1.5}$$

The aim of this paper, is to introduce two new inequalities regarding Riemann-Stieltjes integrals for functions of bounded p-variation and Hölder continuous integrators.

2. Two inequalities for Riemann–Stieltjes integrals

We begin with the following lemma due to L.C. Young:

Lemma 2.1. *([6]) If $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are two ordered set of complex numbers, then the inequality*

$$\sum_{k=1}^{n} |a_k b_k| \le \sum_{k=1}^{n} k^{-\left(\frac{1}{p}+\frac{1}{q}\right)} \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q \right)^{1/q}, \tag{2.1}$$

valid for $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} \ge 1$.

In particular, if $b_i = 1$ for all $i = 1, \cdots, n$, then

$$\sum_{k=1}^{n} |a_k| \le n^{1/q} \sum_{k=1}^{n} k^{-\left(\frac{1}{p}+\frac{1}{q}\right)} \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p}. \tag{2.2}$$

We begin with the following generalization of Theorem 1.2 to functions of bounded p-variations:

Theorem 2.2. *Fix $1 \le p < \infty$. Let $f, g : [a,b] \to \mathbb{R}$ be such that f is a nonconstant continuous function on $[a,b]$ and g is of bounded p–variation on $[a,b]$. Then the Riemann–Stieltjes integral $\int_a^b f(t)\,dg(t)$ exists and the inequality:*

$$\left| \int_a^b f(t)\,dg(t) \right| \le \sup_{t\in[a,b]} |f(t)| \cdot \bigvee_a^b (g;p), \tag{2.3}$$

holds.

Proof. The existence of $\int_a^b f(t)\,dg(t)$, follows trivially. To prove (2.3), assume $p = 1$. Let $\delta_n : a = x_0^{(n)} < x_1^{(n)} < \cdots < x_{n-1}^{(n)} < x_n^{(n)} = b$, is a sequence of divisions, with $\nu(\delta_n) \to 0$ as $n \to \infty$, where $\nu(\delta_n) := \max_{i\in\{0,1\dots,n-1\}} \left(x_{i+1}^{(n)} - x_i^{(n)}\right)$ and $\xi_i^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)}\right]$. Then

$$\begin{aligned}
\left| \int_a^b f(t)\,dg(t) \right| &\le \left| \lim_{\nu(\delta_n)\to 0} \sum_{i=0}^{n-1} f\left(\xi_i^{(n)}\right) \left[g\left(x_{i+1}^{(n)}\right) - g\left(x_i^{(n)}\right)\right] \right| \\
&\le \lim_{\nu(\delta_n)\to 0} \sum_{i=0}^{n-1} \left| f\left(\xi_i^{(n)}\right)\right| \left| g\left(x_{i+1}^{(n)}\right) - g\left(x_i^{(n)}\right)\right| \\
&= \sup_{t\in[a,b]} |f(t)| \cdot \sup_{\delta_n} \sum_{i=0}^{n-1} \left| g\left(x_{i+1}^{(n)}\right) - g\left(x_i^{(n)}\right)\right| \\
&= \sup_{t\in[a,b]} |f(t)| \cdot \bigvee_a^b (g;1).
\end{aligned}$$

Now, assume $p > 1$. Since $f(t)$ is a nonconstant function for all t in $[a,b]$, then by applying the discrete Hölder inequality, we have

$$\begin{aligned}
\left| \int_a^b f(t)\,dg(t) \right| &\le \left| \lim_{\nu(\delta_n)\to 0} \sum_{i=0}^{n-1} f\left(\xi_i^{(n)}\right) \left[g\left(x_{i+1}^{(n)}\right) - g\left(x_i^{(n)}\right)\right] \right| \\
&\le \lim_{\nu(\delta_n)\to 0} \sum_{i=0}^{n-1} \left| f\left(\xi_i^{(n)}\right)\right| \left| g\left(x_{i+1}^{(n)}\right) - g\left(x_i^{(n)}\right)\right| \\
&\le \lim_{\nu(\delta_n)\to 0} \left(\sum_{i=0}^{n-1} \left| f\left(\xi_i^{(n)}\right)\right|^q \right)^{\frac{1}{q}} \left(\sum_{i=0}^{n-1} \left| g\left(x_{i+1}^{(n)}\right) - g\left(x_i^{(n)}\right)\right|^p \right)^{\frac{1}{p}} \\
&\le \sup_{t\in[a,b]} \left(|f(t)|^q\right)^{\frac{1}{q}} \cdot \sup_{\delta_n} \left(\sum_{i=0}^{n-1} \left| g\left(x_{i+1}^{(n)}\right) - g\left(x_i^{(n)}\right)\right|^p \right)^{\frac{1}{p}} \\
&= \sup_{t\in[a,b]} |f(t)| \cdot \bigvee_a^b (g;p),
\end{aligned}$$

where $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, which completes the proof. □

So that we may deduce the following result:

Corollary 2.3. *Let $1 \le p < \infty$. Let $f,g : [a,b] \to \mathbb{R}$ be such that f is continuous on $[a,b]$ and g is $\frac{1}{p}$–H–Hölder continuous on $[a,b]$. Then the Riemann-Stieltjes integral $\int_a^b f(t)\,dg(t)$ exists and we have the following inequality:*

$$\left|\int_a^b f(t)\,dg(t)\right| \le H(b-a)^{1/p} \cdot \sup_{t\in[a,b]} |f(t)|. \tag{2.4}$$

Proof. If g has the α-Hölder property, then g has bounded p-variation with $p = 1/\alpha$. Moreover,

$$\bigvee_a^b (g;p) \le H(b-a)^{1/p},$$

which gives the required result by (2.3). □

We may refine the inequality (2.4), as follows:

Theorem 2.4. *Let $1 \le p < \infty$. Let $f,g : [a,b] \to \mathbb{R}$ be such that is f is a nonconstant bounded function on $[a,b]$, i.e., $\|f\|_\infty := \sup_{t\in[a,b]} |f(t)| < \infty$ exists, with $f \in L^1[a,b]$, and g is $\frac{1}{p}$-H–Holder continuous on $[a,b]$. Then we have the following inequality:*

$$\left|\int_a^b f(t)\,dg(t)\right| \le H\|f\|_\infty^{1-\frac{1}{p}} \cdot \|f\|_1^{\frac{1}{p}}. \tag{2.5}$$

Provided that the Riemann-Stieltjes integral $\int_a^b f(t)\,dg(t)$ exists.

Proof. Assume $p \ge 1$. Let $\delta_n : a = x_0^{(n)} < x_1^{(n)} < \cdots < x_{n-1}^{(n)} < x_n^{(n)} = b$, is a sequence of divisions, with $\nu(\delta_n) \to 0$ as $n \to \infty$, where $\nu(\delta_n) := \max_{i\in\{0,1\ldots,n-1\}} \left(x_{i+1}^{(n)} - x_i^{(n)}\right)$ and $\xi_i^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)}\right]$. Since f is a nonconstant on $[a,b]$, then we can apply the discrete power-mean inequality, as follows:

$$\begin{aligned}
\left|\int_a^b f(t)\,dg(t)\right| &\le \left|\lim_{\nu(\delta_n)\to 0} \sum_{i=0}^{n-1} f\left(\xi_i^{(n)}\right)\left[g\left(x_{i+1}^{(n)}\right) - g\left(x_i^{(n)}\right)\right]\right| \\
&\le \lim_{\nu(\delta_n)\to 0} \sum_{i=0}^{n-1} \left|f\left(\xi_i^{(n)}\right)\right| \left|g\left(x_{i+1}^{(n)}\right) - g\left(x_i^{(n)}\right)\right| \\
&\le \lim_{\nu(\delta_n)\to 0} \left(\sum_{i=0}^{n-1} \left|f\left(\xi_i^{(n)}\right)\right|\right)^{1-\frac{1}{p}} \left(\sum_{i=0}^{n-1} \left|f\left(\xi_i^{(n)}\right)\right| \left|g\left(x_{i+1}^{(n)}\right) - g\left(x_i^{(n)}\right)\right|^p\right)^{\frac{1}{p}},
\end{aligned}$$

and since g is $\frac{1}{p}$–Hölder continuous $p \ge 1$, then there exists $H > 0$ such that

$$\left|g\left(x_{i+1}^{(n)}\right) - g\left(x_i^{(n)}\right)\right| \le H\left|x_{i+1}^{(n)} - x_i^{(n)}\right|^{1/p},$$

it follows that

$$\left|g\left(x_{i+1}^{(n)}\right) - g\left(x_i^{(n)}\right)\right|^p \le H^p\left|x_{i+1}^{(n)} - x_i^{(n)}\right|,$$

therefore, we have

$$\left|\int_a^b f(t)\,dg(t)\right| \le \lim_{\nu(\delta_n)\to 0}\left(\sum_{i=0}^{n-1}\left|f\left(\xi_i^{(n)}\right)\right|\right)^{1-\frac{1}{p}}\left(\sum_{i=0}^{n-1}\left|f\left(\xi_i^{(n)}\right)\right|\left|g\left(x_{i+1}^{(n)}\right)-g\left(x_i^{(n)}\right)\right|^p\right)^{\frac{1}{p}}$$

$$\le H\lim_{\nu(\delta_n)\to 0}\left(\sum_{i=0}^{n-1}\left|f\left(\xi_i^{(n)}\right)\right|\right)^{1-\frac{1}{p}}\left(\sum_{i=0}^{n-1}\left|f\left(\xi_i^{(n)}\right)\right|\left|x_{i+1}^{(n)}-x_i^{(n)}\right|\right)^{\frac{1}{p}}$$

$$= H\left[\sup_{t\in[a,b]}|f(t)|\right]^{1-\frac{1}{p}}\cdot\left(\int_a^b|f(t)|\,dt\right)^{\frac{1}{p}}$$

$$= H\,\|f\|_\infty^{1-\frac{1}{p}}\cdot\|f\|_1^{\frac{1}{p}}$$

which completes the proof. □

3. Proposed Problem

It is well-known that for a Riemann integrable function $p:[a,b]\to\mathbb{R}$ and L–Lipschitzian function $\nu:[a,b]\to\mathbb{R}$, one has the inequality

$$\left|\int_a^b p(t)\,d\nu(t)\right| \le L\int_a^b |p(t)|\,dt. \tag{3.1}$$

Next we propose a generalization of (3.1), as follows:

Conjecture 3.1. *If $w:[a,b]\to\mathbb{R}$ belongs to $L^p[a,b]$, $1\le p,q<\infty$ and $\nu:[a,b]\to\mathbb{R}$ is Hölder continuous mapping of order $\frac{1}{q}$, where $H>0$ is given with $\frac{1}{p}+\frac{1}{q}\ge 1$. Then,*

1. *$\int_a^b w(t)\,d\nu(t)$ exists.*
2. *The inequality*

$$\left|\int_a^b w(t)\,d\nu(t)\right| \le H\cdot C(p,q)\cdot\|w\|_p, \tag{3.2}$$

holds, for all $1\le p<\infty$, where, $\|w\|_p=\left(\int_a^b|w(t)|^p\,dt\right)^{1/p}$, $p\ge 1$ and $C(p,q)$ is a constant of p,q.

3. *What is the best possible constant $C(p,q)$ would satisfies the inequality?.*

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Competing Interests

The author declare no competing interests.

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