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# Fixed Point Results for Generalized Contraction with Application of Biomedical Sciences

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## Abstract

In this article, we find the fixed point, fixed ellipse, and fixed elliptic disc theorems, we introduce the concepts of an orthogonal $\Theta$-contraction, an orthogonal $\Theta_g$-weak contraction, an orthogonal $\Psi_g$-weak JS-contraction, an integral-type orthogonal $\Theta_g$-weak contraction, and an integral-type orthogonal $\Psi_g$-weak JS-contraction. To further prove the validity of the postulates, we validate these using illustrated cases with geometric meanings. The reason for this work is the possibility of a geometric shape like a circle, ellipse, disc, or elliptic disc being among the set of non-unique fixed points. We provide an orthogonal $\Theta$-contraction application to chemical sciences.

***Keywords:*** Common fixed point theorems; Orthogonal metric spaces (OMS); Orthogonal $\Theta$-contraction; Orthogonal $\Theta_g$-weak contraction; Orthogonal $\Psi_g$-weak JS-contraction

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## 1. Introduction and Preliminaries

By examining its applications, one can catch a glimpse of fixed-point theory's breadth in several areas. According to several fixed-point theorems, the functions must have at least one fixed point. We can observe that these findings are often advantageous in the field of mathematics and are essential for determining the existence and singularity of solutions to various mathematical models. The Banach and Caccioppoli fixed-point theorem, which was started by Banach [7] in 1922 and proved by Caccioppoli [9] in 1931, was formed after some scientists established various conditions to discover fixed points. A fixed point must exist for the function if it seizes, according to the Banach and Caccioppoli fixed-point theorem. After this incredible outcome Branciari [6] proved the fixed-point theorem of the Banach-Caccioppoli theorem for a class of generalized metric spaces. Theta-contraction mappings were given a new definition by Jleli and Samet [14] in 2014, and they established a number of fixed-point theorems for them in complete metric spaces (CMS). Fixed-point theorems for $\alpha$-$\psi$ -contractive maps were proven by Samet et al. [19]. Fixed-point results for generalized $\Theta$-contractions were demonstrated by Ahmad et al. [2]. By combining generalised contraction with triangular $\alpha$-orbital acceptable mappings in the sense of Branciari metric spaces, Arshad et al. [5] demonstrated certain fixed point results.

On the other hand, Gordji et al. [13] introduced the concept of an orthogonal set (OS) and generalized the Banach Fixed Point (FP) theorem. Further, fixed point results on orthogonal (generalized) metric spaces have been provided by, Javed et al. [16], and Uddin et al. [22, 23] initiated the notion of an orthogonal structure and established the Banach contraction principle. Aydi et al. [24, 25] established modified $F$-contractions via $\alpha$-admissible mappings and generalized admissible-Meir-Keeler-contractions in the context of generalized metric spaces. For more information see [26-34].

A prominent field of research is the study of geometry of the collection of non-unique fixed points on a map. Suppose that, examine a self-map $\mathcal{M}$ on a metric space (M.S) $(\Xi, d)$ with usual M.S the two-dimensional plan $\mathbb{R}^2$ as:

$$\mathcal{M}(\varpi, \mathfrak{d}) = \begin{cases} (\varpi, \mathfrak{d}), & (\varpi, \mathfrak{d}) \in \varpi^2 + \mathfrak{d}^2 = 1, \\ (1,0) & \text{otherwise} \end{cases}$$

Notice that, the set of non-unique fixed points $\{(\cos n\theta, \sin\ n\theta): n \in \mathbb{Z}, \theta \in [0, 2\pi)\}$ includes the circle $\mathfrak{S}\big((0,0), 1\big)$ centered at $(0,0)$ having radius 1; that is, $\mathfrak{S}\big((0,0), 1\big)$ is a fixed circle of $\mathcal{M}$. For example,

$$\mathcal{M}(\varpi, \mathfrak{d}) = \left(\frac{\varpi}{\varpi^2, \mathfrak{d}^2}, \frac{v}{\varpi^2, \mathfrak{d}^2}\right), \varpi, \mathfrak{d} \in \mathbb{R}$$

Then, $\mathcal{M}\mathfrak{S}(0,1) = \mathfrak{S}(0,1)$, but map $\mathcal{M}$ fixes only two points $(1,0)$ and $(-1,0)$ of the circle $\mathfrak{S}(0,1)$. The purpose of this work is to show the notions of an orthogonal $\Theta$-contraction, an orthogonal $\Theta_g$-weak contraction, a $\Psi_g$-weak JS-contraction and a generalized integral-type $\Theta_g$-weak contraction.

**Definition 1.1** [13] Let $\Xi \neq \emptyset$ be a set and $\perp$ be a binary relation on $\Xi \times \Xi$. If $\exists\, \beta_1 \in \Xi$ such that the following condition holds:

$$(\text{ for all } \delta \in \Xi\ \beta_1 \perp \delta) \text{or } (\text{ for all } \delta \in \Xi\ \delta \perp \beta_1),$$

then the element $\beta_1$ is said to be an O-element and $\Xi$ is an OS.

**Definition 1.2** [13] Let $(\Xi, \perp)$ be an OS and $(\Xi, d)$ be a metric space. Then $(\Xi, \perp, d)$ be an OMS.

**Definition 1.3** [13] Let $(\Xi, \perp)$ be an OS. A sequence $\{\beta_n\}$ is said to be an orthogonal sequence (O-Sequence) if

$$(\forall\, n \in \mathbb{N}, \beta_n \perp \beta_{n+1}) \text{ or } (\forall\, n \in \mathbb{N}, \beta_{n+1} \perp \beta_n).$$

Likewise, a Cauchy sequence $\{\beta_n\}$ is called a Cauchy O-sequence if

$$(\forall\, n \in \mathbb{N}, \beta_n \perp \beta_{n+1}) \text{ or } (\forall\, n \in \mathbb{N}, \beta_{n+1} \perp \beta_n).$$

**Definition 1.4** [13] Suppose $(\Xi, \perp)$ be an OS. A mapping $\mathcal{M}_\perp : \Xi \to \Xi$ is called an orthogonal preserving (O-Preserving) if $\mathcal{M}_\perp \beta \perp \mathcal{M}_\perp \delta$ whence $\beta \perp \delta$.

**Definition 1.5** [13] Let $(\Xi, \perp, d)$ be an OMS. Then $\mathcal{M}_\perp : \Xi \to \Xi$ is called an orthogonal continuous (O-continuous) at $\beta \in \Xi$ if, for each O-sequence $\{\beta_n\}$ in $\Xi$ with $\{\beta_n\} \to \beta$, we have $\mathcal{M}_\perp \beta_n \to \mathcal{M}_\perp \beta$. Also, $\mathcal{M}_\perp$ is said to be O-continuous on $\Xi$ if, $\mathcal{M}_\perp$ is O-continuous at each $\beta \in \Xi$.

**Definition 1.6** [13] Let $(\Xi, \perp, d)$ be an OMS. Then $\Xi$ is said to be an OCMS if every Cauchy O-sequence is convergent in $\Xi$.

**Definition 1.7:** [35] A metric is a non-empty set $\Xi$ a function $d: \Xi \times \Xi \to \mathbb{R}^+$ fulfilling

(i) $d(\varpi, \mathfrak{d}) = 0$ iff $\varpi = \mathfrak{d}$

(ii) $d(\varpi, \mathfrak{d}) = d(\mathfrak{d}, \varpi)$

(iii) $d(\varpi, \mathfrak{d}) \leq d(\varpi, \mathfrak{M}) + d(\mathfrak{M}, \mathfrak{d}), \varpi, \mathfrak{d}, \mathfrak{M} \in \Xi$

**Definition 1.8:** [36] An ellipse and foci at $c_1$, and $c_2$ in a MS $(\Xi, d)$ is given as:

$$g(c_1, c_2, \mathfrak{a}) = \{\varpi \in \Xi : d(c_1, \varpi) + d(c_2, \varpi) = \mathfrak{a}, c_1, c_2 \in \Xi, \mathfrak{a} \in [0, \infty)\}$$

The midpoint $\mathfrak{S}$ of a line $c_1 c_2$ is known as center of an ellipse. The distance:

$$f = (1/2) d(c_1, c_2)$$

is the linear eccentricity ; that is,

$$e = d(c_1, c_2)/\mathfrak{a}.$$

**Example 1.1:** Suppose $\varXi = \mathbb{R}$ and metric $\Pi: \varXi \times \varXi \to \mathbb{R}^+$ described $d(\varpi, \mathfrak{d}) = |\varpi - \mathfrak{d}|, \varpi, \mathfrak{d} \in \varXi$; then, $g(5,10,8) = \{\varpi \in \mathcal{M}: d(5, \varpi) + d(10, \varpi) = 8\}$

$$\{\varpi \in \varXi: |5 - \varpi| + |10 - \varpi| = 8\} = \{3.5, 11.5\}$$

**Definition 1.9:** [15] Let $\Omega$ indicate the functions $\Theta: (0, \infty) \to (1, \infty)$ such that

$(\Theta_1)$: $\Theta$ is non-decreasing;

$(\Theta_3)$: $\exists\, \alpha \in (0,1)$ and $\beta \in (0, \infty]$ s.t $\lim_{\varpi \to 0^+} (\Theta(\mathrm{u}) - 1)/\varpi^{\alpha} = \beta$.

**Definition 1.10:** [36] Let $g(c_1, c_2, \mathfrak{a})$ be an ellipse the foci at $c_1$ and $c_2$ in a MS $(\varXi, d)$. So, $g(c_1, c_2, \mathfrak{a})$ is called fixed ellipse of $\mathcal{M}: \varXi \to \varXi$ if $\mathcal{M}\varpi = \varpi, \varpi \in g(c_1, c_2, \mathfrak{a})$, $\mathfrak{a} \in [0, \infty)$.

## 2. Main Results

We are working with maps that satisfy innovative orthogonal contractions that, under certain circumstances, fix one element in space and produce a set of non-unique fixed points, some of which be geometric objects like ellipses or elliptic discs:

$$d(\mathcal{M}\varpi, \mathcal{M}\mathfrak{d}) > 0 \implies \Theta(d(\mathcal{M}\varpi, \mathcal{M}\mathfrak{d}) \leq \left[\Theta\big(\mathfrak{L}(\varpi, \mathfrak{d})\big)\right]^{\alpha}$$

Where $\mathfrak{L}(\varpi, \mathfrak{d}) = \max\{d(\varpi, \mathfrak{d}), \gamma d(\varpi, \mathcal{M}\varpi) + (1-\gamma) d(\mathfrak{d}, \mathcal{M}\mathfrak{d}), (1-\gamma) d(\varpi, \mathcal{M}\varpi) + \gamma d(\mathfrak{d}, \mathcal{M}\mathfrak{d})$,

$$\gamma d(\varpi, \mathcal{M}\mathfrak{d}) + (1-\gamma) d(\mathfrak{d}, \mathcal{M}\varpi), (1-\gamma) d(\varpi, \mathcal{M}\mathfrak{d}) + \gamma d(\mathfrak{d}, \mathcal{M}\varpi)\},$$

$$\gamma \in [0,1), \alpha \in (0,1), \forall\, \varpi, \mathfrak{d} \in \varXi \text{ with } \varpi \perp \mathfrak{d}$$

**Theorem 2.1**. Let $(\varXi, \perp d)$ be an OCMS and O- map $\mathcal{M}: \varXi \to \varXi$ be a $\perp$-continuous O-preserving orthogonal $\Theta$ contraction. Then, $\mathcal{M}$ has a unique fixed point.

Proof: Define a Picard sequence $\{\varpi_n\} \subseteq \Xi$, and $\varpi_n \perp \mathcal{M}\varpi_n$ or $\mathcal{M}\varpi_n \perp \varpi_n$ $\varpi_{n+1} = \mathcal{M}\varpi_n$, $n \in \mathbb{N}_0$, that initial point $\varpi_0 \in \varXi\ \forall\, \mathfrak{d} \in \varXi\ \varpi_0 \perp \mathfrak{d}$ or $\mathfrak{d} \in \varXi \perp \varpi_0$ If $n \in \mathbb{N}$, $\mathcal{M}^n\varpi = \mathcal{M}^{n+1}\varpi$, and $\mathcal{M}^n\varpi \perp \zeta^{n+1}\varpi$ or $\mathcal{M}^{n+1}\varpi \perp \mathcal{M}^n\varpi$ since $\mathcal{M}$ is O-preserving and $\mathcal{M}^n\varpi$ is a fixed point of $\mathcal{M}$.

$$\Theta\big(\mathrm{d}(\mathcal{M}^n\varpi, \mathcal{M}^{n+1}\varpi)\big) \leq \Theta\big(\mathfrak{L}(\mathcal{M}^{n-1}\varpi, \mathcal{M}^n\varpi)\big),$$

Where

$$\mathfrak{L}(\mathcal{M}^{n-1}\varpi, \mathcal{M}^n\varpi) = \max\{d(\mathcal{M}^{n-1}\varpi, \mathcal{M}^n\varpi), \gamma d(\mathcal{M}^{n-1}\varpi, \mathcal{M}^n\varpi)$$

$$+(1-\gamma) d(\mathcal{M}^n\varpi, \mathcal{M}^{n+1}\varpi), (1-\gamma) d(\mathcal{M}^{n-1}\varpi, \mathcal{M}^n\varpi)$$

$$+\gamma d(\mathcal{M}^{n}\varpi, \mathcal{M}^{n+1}\varpi), \gamma d(\mathcal{M}^{n-1}\varpi, \mathcal{M}^{n+1}\varpi)$$

$$+(1-\gamma)d(\mathcal{M}^{n}\varpi, \mathcal{M}^{n}\varpi), (1-\gamma)d(\mathcal{M}^{n-1}\varpi, \mathcal{M}^{n+1}\varpi)$$

$$+\gamma d(\mathcal{M}^{n}\varpi, \mathcal{M}^{n}\varpi)\}$$

$$= \max\{d(\mathcal{M}^{n-1}\varpi, \mathcal{M}^{n}\varpi), \gamma d(\mathcal{M}^{n-1}\varpi, \mathcal{M}^{n}\varpi)$$

$$+(1-\gamma)d(\mathcal{M}^{n}\varpi, \mathcal{M}^{n+1}\varpi), (1-\gamma)d(\mathcal{M}^{n-1}\varpi, \mathcal{M}^{n}\varpi)$$

$$+\gamma d(\mathcal{M}^{n}\varpi, \mathcal{M}^{n+1}\varpi), \gamma d(\mathcal{M}^{n-1}\varpi, \mathcal{M}^{n+1}\varpi),$$

$$(1-\gamma)d(\mathcal{M}^{n-1}\varpi, \mathcal{M}^{n+1}\varpi)\}$$

Case 1. If $d(\mathcal{M}^{n-1}, \mathcal{M}^{n}\varpi) \leq d(\mathcal{M}^{n}\varpi, \mathcal{M}^{n+1}\varpi)$, then

$$\mathfrak{L}(\mathcal{M}^{n-1}\varpi, \mathcal{M}^{n}\varpi) = d(\mathcal{M}^{n}\varpi, \mathcal{M}^{n+1}\varpi)$$

That is, $$\Theta\big(d(\mathcal{M}^{n}\varpi, \mathcal{M}^{n+1}\varpi)\big) \leq \big[\Theta\big(d(\mathcal{M}^{n}\varpi, \mathcal{M}^{n+1}\varpi)\big)\big]^{\alpha},$$

$\alpha \in (0,1)$, a contradiction.

Case 2. If $d(\mathcal{M}^{n-1}, \mathcal{M}^{n}\varpi) \geq d(\mathcal{M}^{n}\varpi, \mathcal{M}^{n+1}\varpi)$, then

$$\mathfrak{L}(\mathcal{M}^{n-1}\varpi, \mathcal{M}^{n}\varpi) = d(\mathcal{M}^{n}\varpi, \mathcal{M}^{n+1}\varpi)$$

That is, $$\Theta\big(d(\mathcal{M}^{n}\varpi, \mathcal{M}^{n+1}\varpi)\big) \leq \big[\Theta\big(d(\mathcal{M}^{n}\varpi, \mathcal{M}^{n+1}\varpi)\big)\big]^{\alpha}.$$

Following a similar pattern,

$$c\big(d(\mathcal{M}^{n}\varpi, \mathcal{M}^{n+1}\varpi)\big) \leq \big[\Theta\big(d(\mathcal{M}^{n-1}\varpi, \mathcal{M}^{n}\varpi)\big)\big]^{\alpha} \ldots$$

$$\leq \big[\Theta\big(d(\varpi, \mathcal{M}\varpi)\big)\big]^{\alpha} \to 1, \text{as } n \to \infty$$

Using $(\Theta_2)$, $\lim_{n\to\infty} d(\varpi, \mathcal{M}\varpi) = 0$. And $(\Theta_3)$, there exist $\beta \in (0, \infty)$ such that

$$\lim_{n\to\infty} \frac{\Theta\big(d(\mathcal{M}^{n}\varpi, \mathcal{M}^{n+1}\varpi)\big) - 1}{\big(d(\mathcal{M}^{n}\varpi, \mathcal{M}^{n+1}\varpi)\big)^{\alpha}} = \beta$$

If $\beta \in (0, \infty)$ then for $\varepsilon_1 = \beta/4 > 0$, there exists $N_1 > 0$ such that

$$\left|\frac{\Theta\big(d(\mathcal{M}^{n}\varpi, \mathcal{M}^{n+1}\varpi)\big) - 1}{\big(d(\mathcal{M}^{n}\varpi, \mathcal{M}^{n+1}\varpi)\big)^{\alpha}} - \beta\right| < \varepsilon, n \geq N_1$$

implies

$$\frac{\Theta\big(d(\mathcal{M}^n\varpi,\mathcal{M}^{n+1}\varpi)\big)-1}{\big(d(\mathcal{M}^n\varpi,\mathcal{M}^{n+1}\varpi)\big)^{\alpha}}>\beta-\varepsilon_1$$

$$=\frac{3}{4}\beta>\varepsilon_1,n\geq N_1.$$

That is,

$$\big(d(\mathcal{M}^n\varpi,\mathcal{M}^{n+1}\varpi)\big)^{\alpha}<(1/\varepsilon_1)\big(\,\Theta\big(d(\mathcal{M}^n\varpi,\mathcal{M}^{n+1}\varpi)\big)-1\big),n\geq N_1.$$

If $\beta=\infty$, then for any $\varepsilon_2>0$, there exists $N_2>0$ such that

$$\frac{\Theta\big(d(\mathcal{M}^n\varpi,\mathcal{M}^{n+1}\varpi)\big)-1}{\big(d(\mathcal{M}^n\varpi,\mathcal{M}^{n+1}\varpi)\big)^{\alpha}}>\varepsilon_2,n\geq N_2$$

That is,

$$\big(d(\mathcal{M}^n\varpi,\mathcal{M}^{n+1}\varpi)\big)^{\alpha}<(1/\varepsilon_2)\big(\,\Theta\big(d(\mathcal{M}^n\varpi,\mathcal{M}^{n+1}\varpi)\big)-1\big),n>N_2$$

Thus, for all $\beta\in(0,\infty]$ and $\mu=\max\{1/\varepsilon_1,1/\varepsilon_2\}$, there exists $N=\max\{N_1,N_2\}$ such that

$$\big(d(\mathcal{M}^n\varpi,\mathcal{M}^{n+1}\varpi)\big)^{\alpha}<\mu\big(\,\Theta\big(d(\mathcal{M}^n\varpi,\mathcal{M}^{n+1}\varpi)\big)-1\big),n>N.$$

$$\leq\ \mu\big[\Theta\big(d(\mathrm{u},\mathcal{M}\varpi)\big)\big]-1\to 0,as\ n\to\infty$$

That is, $\lim_{n\to\infty}\big(d(\mathcal{M}^n\varpi,\mathcal{M}^{n+1}\varpi)\big)^{\alpha}=0\ \ \exists\, n\geq N$ s.t

$$d(\mathcal{M}^n\varpi,\mathcal{M}^{n+1}\varpi)\leq\frac{1}{n^{1/\alpha}},n\geq N$$

If $n>m$ , $d(\mathcal{M}^m\varpi,\mathcal{M}^n\varpi)\leq d(\mathcal{M}^m\varpi,\mathcal{M}^{m+1}\varpi)+d(\mathcal{M}^{m+1}\varpi,\mathcal{M}^{m+2}\varpi)+\cdots+$ $d(\mathcal{M}^{n-1}\varpi,\mathcal{M}^n\varpi)$

$$\leq\frac{1}{m^{1/\alpha}}+\frac{1}{(m+1)^{1/\alpha}}+\cdots+\frac{1}{(n-1)^{1/\alpha}}\leq\sum_{i=n}^{\infty}\frac{1}{i^{1/\alpha}}$$

Since $\alpha\in(0,1)$, series $\sum_{i=n}^{\infty}\frac{1}{i^{1/\alpha}}$ is convergent and $\lim_{n,m\to\infty}d(\mathcal{M}^m\varpi,\mathcal{M}^n\varpi)$ is finite i.e $\{\mathcal{M}^n\varpi\}$ is a Cauchy orthogonal sequence (COS). So, $\Xi$ is complete, $\{\mathcal{M}^n\varpi\}$ converges to $\varpi^*\in\Xi$ . since $\mathcal{M}$ is continuous,

$$\{\mathcal{M}^n\varpi\}\to\varpi^*\Rightarrow\{\mathcal{M}^{n+1}\varpi\}\to\mathcal{M}\varpi^*.$$

And limit $\mathcal{M}\varpi^* = \varpi^*$, i.e, $\varpi^*$ is a fixed point of $\mathcal{M}$. Let $\mathfrak{M}^*$ be other fixed point of $\Xi$ and $\mathfrak{M}^* \perp \varpi^*$ or $\varpi^* \perp \mathfrak{M}^*$, So $d(\mathcal{M}\varpi^*, \mathcal{M}\mathfrak{M}^*) = d(\varpi^*, \mathfrak{M}^*) > 0$. Now,

$$\Theta\big(d(\mathcal{M}\varpi^*, \mathcal{M}\mathfrak{M}^*)\big) \leq \left[\Theta\big(\mathfrak{L}(\varpi^*, \mathfrak{M}^*)\big)\right]^{\alpha}$$

Where

$$\mathfrak{L}(\varpi^*, \mathfrak{M}^*) = \max\{d(\varpi^*, \mathfrak{M}^*), \gamma d(\varpi^*, \mathcal{M}\Xi^*) + (1-\gamma) d(\mathfrak{M}^*, \Xi\mathfrak{M}^*)\}.$$

$$(1-\gamma) d(\varpi^*, \mathcal{M}\varpi^*) + \gamma d(\mathfrak{M}^*, \Xi\mathfrak{M}^*), \gamma d(\varpi^*, \mathcal{M}\mathfrak{M}^*)$$

$$+(1-\gamma) d(\mathfrak{M}^*, \mathcal{M}\varpi^*), (1-\gamma) d(\varpi^*, \mathcal{M}\mathfrak{M}^*) + \gamma d(\mathfrak{M}^*, \mathcal{M}\varpi^*)\},$$

$$= \max\{d(\Xi^*, \mathfrak{M}^*), \gamma d(\varpi^*, \varpi^*) + (1-\gamma) d(\mathfrak{M}^*, \mathfrak{M}^*)\}$$

$$(1-\gamma) d(\varpi^*, \varpi^*) + \gamma d(\mathfrak{M}^*, \mathfrak{M}^*), \gamma d(\varpi^*, \mathfrak{M}^*)$$

$$+(1-\gamma) d(\mathfrak{M}^*, \varpi^*), (1-\gamma) d(\varpi^*, \mathfrak{M}^*) + \gamma d(\mathfrak{M}^*, \varpi^*)\},$$

$$= \max\{d(\varpi^*, \mathfrak{M}^*), \gamma d(\varpi^*, \mathfrak{M}^*) + (1-\gamma) d(\mathfrak{M}^*, \varpi^*)$$

$$(1-\gamma) d(\varpi^*, \mathfrak{M}^*) + \gamma d(\mathfrak{M}^*, \varpi^*)\} = d(\varpi^*, \mathfrak{M}^*).$$

That is,

$$\Theta\big(d(\mathcal{M}\varpi^*, \mathcal{M}\mathfrak{M}^*)\big) \leq \left|\Theta\big(d(u^*, \mathfrak{M}^*)\big)\right|^{\alpha} \leq \Theta\big((u^*, \mathfrak{M}^*)\big).$$

That is,

$$\Theta\big(d(\mathrm{u}^*, \mathfrak{M}^*)\big) \leq \Theta\big(d(\mathrm{u}^*, \mathfrak{M}^*)\big),$$

a contradiction. Hence, proved.

**Theorem 2.2:** Let $(\Xi, \perp, d)$ be an OCMS $\mathcal{M}: \Xi \to \Xi$ be a self-Mapping O-preserving O-continuous Ciric-type orthogonal $\Theta$-concentration then, $\mathcal{M}$ has a unique fixed point and the sequence of iterates $\{\mathcal{M}^n\varpi\}$ converges to a fixed point of $\mathcal{M}$in $\Xi$.

**Proof:** The proof obeys the method of theorem 7 on $\gamma = 0$.

**Example 2.**1: Let $\Xi = \{\varpi_n = 2n - 1 : n \in \mathbb{N}\}$ and $d(\varpi, \mathfrak{d}) = |\varpi - \mathfrak{d}|\ \forall\ \varpi, \mathfrak{d} \in \Xi$ with $\varpi \perp \mathfrak{d} \Leftrightarrow \varpi \geq \mathfrak{d}$ be an OCMS. Let $\Theta(\mathrm{t}) = \mathrm{e}^{\mathrm{te}^{\mathrm{t}}} \in \Omega, \gamma = 0$. Describe a self-map $\mathcal{M}: \Xi \to \Xi$ :

$$\mathcal{M}\varpi = \begin{cases} \varpi_1, & \varpi = \varpi_1, \\ \varpi_{n-1}, & \varpi = \varpi_n\ n \geq 2 \end{cases}$$

Then,

$$\mathfrak{L}(\mathrm{u_n},\mathrm{u_1}) = \max\{d(\varpi_n,\varpi_1), d(\varpi_n,\mathcal{M}\varpi_n), d(\varpi_1,\mathcal{M}\varpi_1), d(\varpi_n,\mathcal{M}\varpi_1)\}$$
$$= \max\{d(2n-1,1), d(2n-1,2n-3), 0, d(1,2n-3), d(2n-1,1)\}$$
$$= \max\{|2n-2|, |2|, |2n-4|\} = 2n-2 \geq 2.$$

Now,

$$\lim_{n\to\infty} \frac{d(\mathcal{M}\varpi_n,\mathcal{M}\varpi_1)}{\mathfrak{L}(u_n,u_1)}$$
$$= \lim_{n\to\infty} \frac{|2n-3-1|}{2n-2} = 1, n \geq 2.$$

so, satisfies Ciric-type Θ concentration; i.e

$$d(\mathcal{M}\varpi_n,\mathcal{M}\varpi_m) \neq 0 \Rightarrow e^{\sqrt{d(\mathcal{M}\varpi_n,\mathcal{M}\varpi_m)e^{d(\mathcal{M}\varpi_n,\mathcal{M}\varpi_m)}}}$$
$$\leq e^{\alpha\sqrt{d(\varpi_n,\varpi_m)e^{d(\varpi_n,\varpi_m)}}}, \alpha \in (0,1)$$
$$\Rightarrow d(\mathcal{M}\varpi_n,\mathcal{M}\varpi_m)e^{d(\mathcal{M}\varpi_n,\mathcal{M}\varpi_m)}$$
$$\leq \alpha^2\left[d(\varpi_n,\varpi_m)e^{d(\varpi_n,\varpi_m)}\right], \alpha \in (0,1)$$
$$\Rightarrow \frac{d(\mathcal{M}\varpi_n,\mathcal{M}\varpi_m)e^{d(\mathcal{M}\varpi_n,\mathcal{M}\varpi_m)}}{d(\varpi_n,\varpi_m)e^{d(\varpi_n,\varpi_m)}} \leq \alpha^2, \alpha \in (0,1).$$

**Case 1**. When $n = 1$ and $m \geq 2$,

$$\frac{d(\mathcal{M}\varpi_n,\mathcal{M}\varpi_m)e^{d(\mathcal{M}\varpi_n,\mathcal{M}\varpi_m)}}{d(\varpi_n,\varpi_m)e^{d(\varpi_n,\varpi_m)}} = \frac{\left|(2m-4)e^{(2m-4)}\right|}{(2m-2)e^{(2m-2)}} \leq e^{-2}$$

**Case 2**. When $n > m > 1$,

$$\frac{d(\mathcal{M}\varpi_n,\mathcal{M}\varpi_m)e^{d(\mathcal{M}\varpi_n,\mathcal{M}\varpi_m)}}{d(\varpi_n,\varpi_m)e^{d(\varpi_n,\varpi_m)}} = \frac{|2n-2m-6|e^{|2n-2m-6|}}{|2n-2m-2|e^{|2n-2m-2|}}$$
$$\frac{(2n-2m-6)e^{(2n-2m-6)}}{(2n-2m-2)e^{(2n-2m-2)}} \leq e^{-4}$$

Where, $\mathcal{M}$ is Ciric-type Θ concentration with $\alpha = \max\{e^{-4}, e^{-2}\} = e^{-2}$ and unique fixed point 1. and more as $\lim_{n\to\infty} \mathcal{M}^n\varpi_1 = 1$ ..

**Definition 2.1** An elliptic disc have foci at $c_1$ and $c_2$ in a OCMS $(\Xi, \perp, d)$ is describe as $g_{\mathfrak{D}}(c_1,c_2,\mathfrak{a}) = \{\varpi \in \Xi : d(c_1,\varpi) + d(c_2,\varpi) \leq \mathfrak{a}, c_1, c_2 \in \mathcal{M}, \mathfrak{a} \in [0,\infty)\}$ ,

**Definition 2.2:** Let $g_{\mathfrak{D}}(c_1, c_2, \mathfrak{a})$ be an elliptic disc having foci at $c_1$ and $c_2$ in a MS $(\Xi, d)$. $g_{\mathfrak{D}}(c_1, c_2, \mathfrak{a})$ is called fixed elliptic disc of map $\mathcal{M}: \Xi \to \Xi$ if

$$\mathcal{M}\varpi = \varpi, \varpi \in g_{\mathfrak{D}}(c_1, c_2, \mathfrak{a}), \mathfrak{a} \in [0, \infty)$$

**Definition 2.3:** Let $\Theta : (0, \infty) \to (1, \infty)$ be an increasing function. A map $\mathcal{M}: \Xi \to \Xi$ of an OMS $(\Xi, \perp, d)$ is called orthogonal $\Theta_g$-week contraction with $\varpi \neq \mathfrak{d}$, if

$$d(\varpi, \mathcal{M}\varpi) > 0 \Rightarrow \Theta\big(d(\varpi, \mathcal{M}\varpi) \leq \left[\Theta(\mathfrak{L}(\mathrm{u}, \mathfrak{d}))\right]^{\alpha}$$

Where $\mathfrak{L}(\mathrm{u}, \mathfrak{d}) = \max\{d(\mathrm{u}, \mathfrak{d}), \gamma d(\varpi, \mathcal{M}\varpi) + (1-\gamma) d(\mathfrak{d}, \mathcal{M}\mathfrak{d}), (1-\gamma) d(\varpi, \mathcal{M}\varpi)$

$$+\gamma d(\mathfrak{d}, \mathcal{M}\mathfrak{d}), \gamma d(\varpi, \mathcal{M}\mathfrak{d}) + (1-\gamma) d(\mathfrak{d}, \mathcal{M}\varpi), (1-\gamma) d(\varpi, \mathcal{M}\mathfrak{d}) + \gamma d(\mathfrak{d}, \mathcal{M}\varpi),$$

$\gamma \in [0,1), \alpha \in (0,1), \varpi, \mathfrak{d} \in \Xi$ with $\varpi \perp \mathfrak{d}$

**Theorem 2.3:** Let $g(c_1, c_2, \mathfrak{a})$ be ellipse in an OMS $(\Xi, \perp, d)$ and

$$\mathfrak{a} = \left(\frac{1}{2}\right)\{\inf d(\varpi, \mathcal{M}\varpi): \varpi \neq \mathcal{M}\varpi\}$$

A $\mathcal{M}: \Xi \to \Xi$ is a self-mapping O-preserving O-continuous orthogonal $\Theta_g$-week contraction with $c_1, c_2 \in \Xi$ and $c_1 \perp c_2$ and $d(c_1, \mathcal{M}\varpi) + d(c_2, \mathcal{M}\varpi) = \mathfrak{a}, \varpi \in g(c_1, c_2, \mathfrak{a})$ then $g(c_1, c_2, \mathfrak{a})$ is a fixed ellipse of $\mathcal{M}$.

Proof: Let $\varpi \in g(c_1, c_2, \mathfrak{a})$ be any random point and $\mathcal{M}\varpi \neq \varpi$ *with* $\mathcal{M}\varpi \perp \varpi$ or $\varpi \perp \mathcal{M}\varpi$ $\mathfrak{a}, d(\varpi, \mathcal{M}\varpi) \geq 2\mathfrak{a}$, let $\mathcal{M}c_1 \neq c_1$with $\mathcal{M}c_1 \perp c_1$ or $c_1 \perp \mathcal{M}c_1$ and $\mathcal{M}c_2 \neq c_2$, and $\mathcal{M}c_2 \perp c_2$ or $c_2 \perp \mathcal{M}c_2$so $\mathcal{M}$ is O-preserving we have $d(c_1, \mathcal{M}c_1) > 0$, $d(c_2, \mathcal{M}c_2) > 0$, and

$$\Theta\big(d(c_1, \mathcal{M}c_1)\big) \leq \left[\Theta\big(\mathfrak{L}(c_1, c_2)\big)\right]^{\alpha}$$

$$= [\Theta(\max\{d(c_1, c_1), \gamma d(c_1, \mathcal{M}c_1) + (1-\gamma) d(c_1, \mathcal{M}c_1), (1-\gamma) d(c_1, \mathcal{M}c_1) + \gamma d(c_1, \mathcal{M}c_1) \gamma d(c_1, \mathcal{M}c_1) + (1-\gamma) d(c_1, \mathcal{M}c_1), (1-\gamma) d(c_1, \mathcal{M}c_1) + \gamma d(c_1, \mathcal{M}c_1)\})]^{\alpha}$$

$$= [\Theta(\max\{0, \mathrm{d}(c_1, \mathcal{M}c_1)\})]^{\alpha} = \left[\Theta\big(d(c_1, \mathcal{M}c_1)\big)\right]^{\alpha} < \Theta\big(d(c_1, \mathcal{M}c_1)\big)$$

$\alpha \in (0,1)$, a contradiction. So $\mathcal{M}c_1 = c_1$. Similarly, $\mathcal{M}c_2 = c_2$ again, since

$$d(\varpi, \mathcal{M}\varpi) > 0, \Theta\big(d(\varpi, \mathcal{M}\varpi)\big) \leq \left[\Theta\big(\mathfrak{L}(\varpi, c_1)\big)\right]^{\alpha}$$

$$= [\Theta(\max\{d(\varpi, c_1), \gamma d(\varpi, \mathcal{M}\varpi) + (1-\gamma) d(c_1, \mathcal{M}c_1), (1-\gamma) d(\varpi, \mathcal{M}\varpi) + \gamma d(c_1, \mathcal{M}c_1) \gamma d(\varpi, \mathcal{M}c_1) + (1-\gamma) d(c_1, \mathcal{M}\varpi), (1-\gamma) d(\varpi, \mathcal{M}c_1) + \gamma d(c_1, \mathcal{M}\varpi)\})]^{\alpha}$$

$$< [\Theta(\max\{2\mathfrak{a}, \gamma d(\varpi, \mathcal{M}\varpi), (1-\gamma) d(\varpi, \mathcal{M}\varpi), \gamma d(\varpi, c_1) + (1-\gamma) d(c_1, \mathcal{M}\varpi), (1-\gamma) d(\varpi, c_1) + \gamma d(c_1, \mathcal{M}\varpi)\})]^{\alpha}$$

$$< \Theta\left(\max\left\{\begin{matrix} 2\mathfrak{a}, \gamma d(\varpi,\mathcal{M}\varpi), (1-\gamma)d(\varpi,\mathcal{M}\varpi), \\ \gamma d(\varpi,c_1)+(1-\gamma)d(c_1,\mathcal{M}\varpi), \\ (1-\gamma)d(\varpi,c_1)+ \quad \gamma d(c_1,\mathcal{M}\varpi) \end{matrix}\right\}\right), \alpha \in (0,1)$$

**Case 1:** If

$$\max\left\{\begin{matrix} 2\mathfrak{a}, \gamma d(\varpi,\mathcal{M}\varpi), (1-\gamma)d(\varpi,\mathcal{M}\varpi), (1-\gamma)d(\varpi,\mathcal{M}\varpi), \\ \gamma d(\varpi,c_1)+(1-\gamma)d(c_1,\mathcal{M}\varpi), (1-\gamma)\, d(\varpi,c_1) \quad +\gamma d(c_1,\mathcal{M}\varpi) \end{matrix}\right\} = 2\mathfrak{a}$$

Then $\Theta(\mathrm{d}(\varpi,\mathcal{M}\varpi) < \Theta(2\mathfrak{a}.)$ By definition of $\mathfrak{a}$ and $\Theta, \Theta(2\mathfrak{a}) \leq \Theta\big(d(\varpi,\mathcal{M}\varpi)\big) < \Theta(2\mathfrak{a})$, a contradiction.

**Case 2**. If

$$\max\left\{\begin{matrix} 2\mathfrak{a}, \gamma d(\varpi,\mathcal{M}\varpi), (1-\gamma)d(\varpi,\mathcal{M}\varpi), (1-\gamma)d(\varpi,\mathcal{M}\varpi), \\ \gamma d(\varpi,c_1)+(1-\gamma)d(c_1,\mathcal{M}\varpi), (1-\gamma)d(\varpi,c_1) \quad +\gamma d(c_1,\mathcal{M}\varpi) \end{matrix}\right\} = \gamma d(\varpi,\mathcal{M}\varpi),$$

Then

$$\gamma d(\varpi,\mathcal{M}\varpi)(\gamma d(\varpi,\mathcal{M}\varpi).$$

If $\gamma = 0, \Theta\big(d(\varpi,\mathcal{M}\varpi)\big) < \Theta(0)$, a contradiction. If $\gamma \in (0,1), \Theta\big(d(\varpi,\mathcal{M}\varpi)\big) < \Theta\big(\gamma d(\varpi,\mathcal{M}\varpi)\big) < \Theta\big(d(\varpi,\mathcal{M}\varpi)\big)$, a contradiction .

**Case 3**. If

$$\max\left\{\begin{matrix} 2\mathfrak{a}, \gamma d(\varpi,\mathcal{M}\varpi), (1-\gamma)d(\varpi,\mathcal{M}\varpi), \\ (1-\gamma)d(\varpi,\mathcal{M}\varpi), \\ \gamma d(\varpi,c_1)+(1-\gamma)d(c_1,\mathcal{M}\varpi), \\ (1-\gamma)d(\varpi,c_1) \quad +\gamma d(c_1,\mathcal{M}\varpi) \end{matrix}\right\} = (1-\gamma)d(\varpi,\mathcal{M}\varpi),$$

Then $\Theta\big(d(\varpi,\mathcal{M}\varpi)\big) < \Theta\big((1-\gamma)d(\varpi,\mathcal{M}\varpi)\big) \leq \Theta\big(d(\varpi,\mathcal{M}\varpi)\big)$, a contradiction.

**Case 4**. If

$$\max\left\{\begin{matrix} 2\mathfrak{a}, \gamma d(\varpi,\mathcal{M}\varpi), (1-\gamma)d(\varpi,\mathcal{M}\varpi) \\ , (1-\gamma)d(\varpi,\mathcal{M}\varpi), \\ \gamma d(\varpi,c_1)+(1-\gamma)d(c_1,\mathcal{M}\varpi), \\ (1-\gamma)d(\varpi,c_1) \quad +\gamma d(c_1,\mathcal{M}\varpi) \end{matrix}\right\} = \gamma d(\varpi,c_1)+(1-\gamma)d(c_1,\mathcal{M}\varpi),$$

Then

$$\begin{aligned} \Theta\big(d(\varpi,\mathcal{M}\varpi)\big) &< \Theta\big(\gamma d(\varpi,c_1)+(1-\gamma)d(c_1,\mathcal{M}\varpi)\big) \\ &< \Theta(\gamma\mathfrak{a}+(1-\gamma)\mathfrak{a}) = \Theta(\mathfrak{a}) \end{aligned}$$

By definition of $\mathfrak{a}$ and $\Theta$, $\Theta(2\mathfrak{a}) \leq \Theta(\mathrm{d}(\varpi,\mathcal{M}\varpi) < \Theta(\mathfrak{a})$, a contradiction.

**Case 5**. If

$$\max\left\{\begin{matrix} 2\mathfrak{a}, \gamma d(\varpi,\mathcal{M}\varpi), (1-\gamma)d(\varpi,\mathcal{M}\varpi), \\ (1-\gamma)d(\varpi,\mathcal{M}\varpi) \\ , \gamma d(\varpi,c_1)+(1-\gamma)d(c_1,\mathcal{M}\varpi), \\ (1-\gamma)d(\varpi,c_1) \quad +\gamma d(c_1,\mathcal{M}\varpi) \end{matrix}\right\} = (1-\gamma)d(\varpi,c_1)+\gamma d(c_1,\mathcal{M}\varpi),$$

Then

$$\Theta\big(d(\varpi,\mathcal{M}\varpi)\big) < \Theta\big((1-\gamma)d(\varpi,c_1)+\gamma d(c_1,\mathcal{M}\varpi)\big)$$

$$< \Theta(\gamma\mathfrak{a} + (1-\gamma)\mathfrak{a}) = \Theta(\mathfrak{a}).$$

By definition of $\mathfrak{a}$ and $\Theta, \Theta(2\mathfrak{a}) \leq \Theta(d(\varpi, \mathcal{M}\varpi) < \Theta(\mathfrak{a})$, Hence, $\mathcal{M}\varpi = \varpi, \varpi \in g(c_1, c_2, \mathfrak{a})$; that is $g(c_1, c_2, \mathfrak{a})$ is a fixed ellipse of $\Xi$.

**Theorem 2.4**: If in the above theorem $\mathrm{d}(c_1, \mathcal{M}\varpi) + d(c_2, \mathcal{M}\varpi) \leq \mathfrak{a}$ , then $g_{\mathfrak{D}}(c_1, c_2, \mathfrak{a})$ is a fixed elliptic disc of $\mathcal{M}$.

**Proof:** We have to prove $g_{\mathfrak{D}}(c_1, c_2, \mathfrak{a})$ is a fixed elliptic disc of $\mathcal{M}$, and that $\mathcal{M}$ fixes an ellipse $g(c_1, c_2, \mathfrak{a})$ with $\mathfrak{b} \lhd \mathfrak{a}$. Since $\mathcal{M}$ is orthogonal $\Theta_g$ weak concentration then $d(c_1, \mathcal{M}\varpi) + d(c_2, \mathcal{M}\varpi) = \mathfrak{b} \leq \mathfrak{a}$; that is $\mathcal{M}\varpi = \varpi, \forall \varpi \in g_{\mathfrak{D}}(c_1, c_2, \mathfrak{b})$. Hence $g_{\mathfrak{D}}(c_1, c_2, \mathfrak{a})$ is a fixed elliptic disc of $\mathcal{M}$.

**Theorem 2.5:** Theorem 15 remains true if we use Ciric type orthogonal $\Theta_g$- weak concentration.

**Proof:** The proof follows the pattern of Theorem 15 on taking $\gamma = 0$.

**Example 2.2:** Let $\Xi = [5, \infty]$ and $d(\varpi, \mathfrak{d}) = |\varpi - \mathfrak{d}| \; \forall \; \varpi, \mathfrak{d} \in \; \Xi$ with $\varpi \perp \mathfrak{d} \Leftrightarrow \varpi \geq \mathfrak{d}$ is an OCMS.

Let $\Theta(\mathrm{t}) = \mathrm{e}^{\mathrm{t}}, c_1 = -2, c_2 = 3$ , $\mathfrak{a} = 6, \gamma = 0$, and $\alpha = 6/7$. An ellipse

$$g(-2,3,6) = \{\varpi \in \Xi : d(-2, \varpi) + d(3, \varpi) = 6\}$$
$$\{\varpi \in \Xi : |-2 - \varpi| + |3 - \varpi = 6\} = \{-2.5, 3.5\}|$$

Describe a self-map $\mathcal{M}: \Xi \to \Xi$ as

$$\mathcal{M}\varpi = \begin{cases} \varpi, & \varpi \in [-5,5], \\ \varpi + 12, & \text{otherwise} \end{cases}$$

Since for $\varpi \in [-5,5], d(\varpi, \mathcal{M}\varpi) = 0$ and for $\varpi \in (5, \infty), d(\varpi, \mathcal{M}\varpi) = 12 > 0$.

**Case 1**. For $\varpi > 5$ and $c_1 = -2$,

$$\mathfrak{L}(u, -2) = \max\{d(\varpi, -2), d\big(-2, \mathcal{M}(-2)\big), d(\varpi, \mathcal{M}\varpi), d(-2, \mathcal{M}\varpi), d(\varpi, \mathcal{M}(-2))\}$$
$$= \max\{d(\varpi, -2), d(-2, -2), d(\varpi, \mathcal{M}\varpi), d(-2, \mathcal{M}\varpi), d(\varpi, -2)\}$$
$$= \max\{d(\varpi, -2), 0, 12, d(-2, \varpi + 12)\} = \max\{|\varpi + 2|, 12, |\varpi + 14|\}$$
$$= |\varpi + 14| > 19,$$

and

$$\Theta\big(d(\varpi, \mathcal{M}\varpi)\big) = \Theta(12) = \mathrm{e}^{12} < e^{(12/13)[\varpi+8]} = e^{\left(\mathfrak{L}(\varpi,-2)\right)^{(12/19)}} = [\Theta(\mathfrak{L}(\varpi, -2))]^{12/19}.$$

**Case 2**. For $\varpi > 5$ and $c_2 = 3$,

$$\mathfrak{L}(\varpi, 3) = \max\{d(\varpi, 3), d(3, \mathcal{M}3), d(\varpi, \mathcal{M}\varpi), d(3, \mathcal{M}\varpi), d(\varpi, \mathcal{M}3)\}$$
$$= \max\{d(\varpi, 3), d(3,3), d(\varpi, \mathcal{M}\varpi), d(3, \mathcal{M}\varpi), d(\varpi, 3)\}$$
$$= \max\{d(\varpi, 3), 0, 12, d(3, \varpi + 12)\}$$
$$= \max\{|\varpi - 3|, |\varpi + 9|\} = |\varpi + 9| > 14,$$

and

$$\Theta\big(d(\varpi, \mathcal{M}\varpi)\big) = \Theta(12) = e^{12} < e^{(12/13)[\varpi+9]}$$
$$= e^{\left(\mathfrak{L}(\varpi,3)\right)^{(12/14)}} = [\Theta(\mathfrak{L}(\varpi, 3))]^{(12/14)}.$$

with $c_1 = -2, c_2 = 3$ and $\alpha = \max\{12/19, 12/14\} = 12/14$. Hence $g(-2,3,6) = \{-2.5, 3.5\}$ is a fixed ellipse and $g_{\mathfrak{D}}(-2,3,6) = [-2.5, 3.5]$ is a fixed elliptic disc of $\mathcal{M}$,

$$d(-2, \varpi) + d(3, \varpi) \leq 6, \varpi \in g_{\mathfrak{D}}(-2,3,6).$$

**Example 2.3:** Suppose $\Xi = \mathbb{R}^2$ and a metric $d: \Xi \times \Xi \to \mathbb{R}^*$ be describe as $d(\varpi, \mathfrak{d}) = \sqrt{(\varpi_1 - \mathfrak{d}_1)^2 + (\varpi_2 - \mathfrak{d}_2)^2}$ , where $\varpi = (\varpi_1, \varpi_2)$ and $\mathfrak{d} = (\mathfrak{d}_1, \mathfrak{d}_2)$. $d(\varpi, \mathfrak{d}) \; \forall \; \varpi, \mathfrak{d} \in \Xi$ with $\varpi \perp \mathfrak{d} \Leftrightarrow \varpi \geq \mathfrak{d}$

Let $\Theta(\mathrm{t}) = 1 + \mathrm{t}, c_1 = \big(3 + 2\sqrt{3}, -1\big), c_2 = \big(3 - 2\sqrt{3}, -1\big), \mathfrak{a} = 8, \gamma = 0$ and $\alpha = 6/7$.

The ellipse

$$g(c_1, c_1, 8) = \{\varpi \in \Xi : d(c_1, \varpi) + d(c_2, \varpi) = 8\}$$

$$\left\{\varpi \in \Xi : \sqrt{\left(\varpi_1 - 3 - 2\sqrt{3}\right)^2 + (\varpi_2 + 1)^2} + \sqrt{\left(\varpi_1 - 3 + 2\sqrt{3}\right)^2 + (\varpi_2 + 1)^2} = 8\right\}$$

$$= \left\{\varpi \in \Xi : \frac{(\varpi_1 - 3)^2}{16} + \frac{(\varpi_2 + 1)^2}{4} = 1\right\},$$

Further, the elliptic disc

$$g_{\mathfrak{D}}(c_1, c_2, 8) = \left\{\varpi \in \Xi : \frac{(\varpi_1 - 3)^2}{16} + \left(\frac{(\varpi_2 + 1)^2}{4}\right) \leq 1\right\}$$

Describe a self-map $\mathcal{M} : \Xi \to \Xi$ as

$$\mathcal{M}\varpi = \begin{cases} \varpi, & \varpi \in (3 + 6\cos\theta, -1 + 6\sin\theta), \\ \varpi + \left(8\sqrt{2}, \quad 8\sqrt{2}\right), & \text{otherwise} \end{cases}$$

Since for $\varpi \in (3 + 6\cos\theta, -1 + 6\sin\theta)$, $d(\varpi, \mathcal{M}\varpi) = 0$ and for $\varpi \in \mathbb{R}^2 \backslash (3 + 6\cos\theta, -1 + 6\sin\theta)$,

$$d(\varpi, \mathcal{M}\varpi) = 16 > 0.$$

**Case 1.** For $\varpi \in \mathbb{R}^2 \backslash (3 + 6\cos\theta, -1 + 6\sin\theta)$ and $c_1 = \left(3 + 2\sqrt{3}, -1\right)$,

$$\mathfrak{L}\left(\varpi, \left(3 + 2\sqrt{3}, -1\right)\right)$$

$$= \max\left\{\begin{matrix} d\left(\varpi, \left(3 + 2\sqrt{3}, -1\right)\right), d\left(\left(3 + 2\sqrt{3}, -1\right), \mathcal{M}\left(3 + 2\sqrt{3}, -1\right)\right), \\ d(\varpi, \mathcal{M}\varpi), d\left(\left(3 + 2\sqrt{3}, -1\right), \mathcal{M}\varpi\right), d\left(\varpi, \mathcal{M}\left(3 + 2\sqrt{3}, -1\right)\right) \end{matrix}\right\}$$

$$= \max\left\{\begin{matrix} d\left(\varpi, \left(+2\sqrt{3}, -1\right)\right), d\left(\left(3 + 2\sqrt{3}, -1\right), \left(3 + 2\sqrt{3}, -1\right)\right), \\ d(\varpi, \mathcal{M}\varpi), d\left(\left(3 + 2\sqrt{3}, -1\right), \mathcal{M}\varpi\right), d\left(\varpi, \left(3 + 2\sqrt{3}, -1\right)\right) \end{matrix}\right\}$$

$$= \max\left\{d\left(\varpi, \left(3 + 2\sqrt{3}, -1\right)\right), 0, 16, d\left(\left(3 + 2\sqrt{3}, -1\right), \varpi + \left(8\sqrt{2}, 8\sqrt{2}\right)\right)\right\}$$

$$= \max\left\{\begin{matrix} \sqrt{\left(\varpi_1 - 3 - 2\sqrt{3}\right)^2 + (\varpi_2 + 1)^2}, 16, \\ \sqrt{\left(\varpi_1 + 8\sqrt{2} - 3 - 2\sqrt{3}\right)^2 + \left(\varpi_2 + 8\sqrt{2} + 1\right)^2} \end{matrix}\right\} > 16$$

and

$$\Theta\left(d(\varpi, \mathcal{M}\varpi)\right) = \Theta(16) = e^{16} < e^{(16/17)\mathfrak{L}\left(\varpi, \left(3 + 2\sqrt{3}, -1\right)\right)} = \left[\Theta\left(\mathfrak{L}\left(\varpi, \left(3 + 2\sqrt{3}, -1\right)\right)\right)\right]^{(16/17)}.$$

**Case 2.** For $\varpi \in \mathbb{R}^2 \backslash (3 + 6\cos\theta, -1 + 6\sin\theta)$ and $c_1 = \left(3 - 2\sqrt{3}, -1\right)$

$$\mathfrak{L}\left(\varpi,\left(3-2\sqrt{3},-1\right)\right)$$
$$=\max\left\{\begin{matrix} d\left(\varpi,\left(3-2\sqrt{3},-1\right),d\left(\left(3-2\sqrt{3},-1\right),\mathcal{M}\left(3-2\sqrt{3},-1\right)\right),\right. \\ d(\varpi,\mathcal{M}\varpi),d\left(\left(3-2\sqrt{3},-1\right),\mathcal{M}\varpi\right),d\left(\varpi,\mathcal{M}\left(3-2\sqrt{3},-1\right)\right)\end{matrix}\right\}$$
$$=\max\left\{\begin{matrix} d\left(\varpi,\left(3-2\sqrt{3},-1\right),d\left(\left(3-2\sqrt{3},-1\right),\left(3-2\sqrt{3},-1\right)\right),\right. \\ d(\varpi,\mathcal{M}\varpi),d\left(\left(3-2\sqrt{3},-1\right),\mathcal{M}\varpi\right),d\left(\varpi,\left(3-2\sqrt{3},-1\right)\right)\end{matrix}\right\}$$
$$=\max\left\{d\left(\varpi,\left(3-2\sqrt{3},-1\right),0,16,d\left(\left(3-2\sqrt{3},-1\right),\varpi+\left(8\sqrt{2},8\sqrt{2}\ \right)\right)\right.\right\}$$
$$=\max\left\{\begin{matrix}\sqrt{\left(\varpi_1-3+2\sqrt{3}\right)^2+(\varpi_2+1)^2}\ ,16, \\ \sqrt{\left(\varpi_1+8\sqrt{2}-3+2\sqrt{3}\right)^2+\left(\varpi_2+8\sqrt{2}+1\right)^2}\end{matrix}\right\}>21$$

And
$$\Theta\left(d(\varpi,\mathcal{M}\varpi)\right)=\Theta(16)=\mathrm{e}^{16}<e^{(16/21)\mathfrak{L}\left(\varpi,\left(3-2\sqrt{3},-1\right)\right)}=\left[\Theta\left(\mathfrak{L}\left(\varpi,\left(3-2\sqrt{3},-1\right)\right)\right)\right]^{(16/21)}$$

i.e, $c_1=\left(3+2\sqrt{3}-1\right),c_2=\left(3-2\sqrt{3}-1\right),\alpha=\max\{16/17\,,16/21\}=16/17.$ Hence $g(c_1,c_2,8)$ is a fixed ellipse and $g_{\mathfrak{D}}(c_1,c_2,8)$ is a fixed elliptic disc of $\mathcal{M}$. That is $d(c_1,\varpi)+d(c_2,\varpi)\le 8,\varpi\in g_{\mathfrak{D}}(c_1,c_2,8)$.

**Definition 2.4:** Let $\Psi:[0,\infty)\to[1,\infty)$ be an increasing function with $\Psi(0)=1$; and $\mathcal{M}:\Xi\to\Xi$ be a self-mapping of an OMS $(\Xi,\perp,d)$ is called an orthogonal $\Psi_g$-weak JS-contraction with $\varpi\neq\mathfrak{d}$, if
$$d(u,\mathcal{M}\varpi)>0\ \Rightarrow\ \Psi\left(d(u,\mathcal{M}\varpi)\right)$$
$$\le\left[\Psi\left(d(u,\mathfrak{d})\right)\right]^a\left[\Psi\left(d(u,\mathcal{M}\varpi)\right)\right]^b\left[\Theta\left(d(\mathfrak{d},\mathcal{M}\mathfrak{d})\right)\right]^c\left[\Psi\left(d(u,\mathcal{M}\mathfrak{d})\right)\right]^e\left[\Psi\left(d(\mathfrak{d},\mathcal{M}\varpi)\right)\right]^f$$

Where $a,b,c.e$ and $f$ are non-negative and $a+b+c+e+f\in[0,1)$, $\varpi,\mathfrak{d}\in\Xi$ with $\varpi\perp\mathfrak{d}$

**Theorem 2.6:** Let $g(c_1,c_2,\mathfrak{a})$ be ellipse in a metric space $(\Xi,d)$ and $\mathfrak{a}=(1/2)\{\inf\ \mathrm{d}(\mathrm{u},\mathcal{M}\varpi):\varpi\neq\mathcal{M}\varpi\}$. The map $\mathcal{M}:\Xi\to\Xi$ is an O-preserving O-continuous and orthogonal $\Psi_g$ -weak JS-contraction with $c_1,c_2\in\Xi$ and $\mathrm{d}(c_1,\mathcal{M}\varpi)+\mathrm{d}(c_2,\mathcal{M}\varpi)=\mathfrak{a},\varpi\in g(c_1,c_2,\mathfrak{a})$, then $g(c_1,c_2,\mathfrak{a})$ is a fixed ellipse of $\Xi$.

**Proof**: Let $\varpi \in g(c_1, c_2, \mathfrak{a})$ be any random point and $\mathcal{M}\varpi \neq \varpi$. with $\mathcal{M}\varpi \perp \varpi$ or $\varpi \perp \mathcal{M}\varpi$ defined by $\mathfrak{a}, d(\varpi, \mathcal{M}\varpi) \geq 2\mathfrak{a}$. Since $\mathcal{M}$ is an orthogonal $\Psi_g$-weak JS-contraction $c_1, c_2 \in \varXi$, suppose $\mathcal{M}c_1 \neq c_1$ and $\mathcal{M}c_2 \neq c_2$, so, we have $\mathrm{d}(c_1, \mathcal{M}c_1) > 0, \mathrm{d}(c_2, \mathcal{M}c_2) > 0$ and

$$\Psi\big(d(c_1, \mathcal{M}c_1)\big)$$
$$\leq \big[\Psi\big(d(c_1, c_1)\big)\big]^a \big[\Psi\big(d(c_1, \mathcal{M}c_1)\big)\big]^b \big[\Psi\big(d(c_1, \mathcal{M}c_1)\big)\big]^c \big[\Psi\big(d(c_1, \mathcal{M}c_1)\big)\big]^e \big[\Psi\big(d(c_1, \mathcal{M}c_1)\big)\big]^f$$

$$= [\Psi(0)]^{a}\big[\Psi\big(d(c_1, \mathcal{M}c_1)\big)\big]^{\mathrm{b+c+e+f}}$$

$$= \big[\Psi\big(d(c_1, \mathcal{M}c_1)\big)\big]^{1-a} < \Psi\big(d(c_1, \mathcal{M}c_1)\big)$$

a contradiction. So $\mathcal{M}c_1 = c_1$. Similarly, $\mathcal{M}c_2 = c_2$. Again, since $d(\varpi, \mathcal{M}\varpi) > 0$, so

$$\Psi\big(d(u, \mathcal{M}\varpi)\big)$$
$$\leq \big[\Psi\big(d(\varpi, c_1)\big)\big]^a \big[\Psi\big(d(\varpi, \mathcal{M}\varpi)\big)\big]^b \big[\Psi\big(d(c_1, \mathcal{M}c_1)\big)\big]^c \big[\Psi\big(d(\varpi, \mathcal{M}c_1)\big)\big]^e \big[\Psi\big(d(c_1, \mathcal{M}\varpi)\big)\big]^f$$

$$< [\Psi(\mathfrak{a})]^a [\Psi(2\mathfrak{a})]^b \big[\Psi\big(d(c_1, c_1)\big)\big]^c [\Psi(\mathfrak{a})]^e [\Psi(\mathfrak{a})]^f$$

$$< [\Theta(2\mathfrak{a})]^a [\Psi(2\mathfrak{a})]^b [\Theta(0)]^c [\Psi(2\mathfrak{a})]^e [\Psi(2\mathfrak{a})]^f.$$

$$[\Psi(2\mathfrak{a})]^{a+b+e+f} < [\Psi(2\mathfrak{a})]^{1-c} < \Psi(2\mathfrak{a})$$

Since $d(\varpi, \mathcal{M}\varpi) \geq 2\mathfrak{a}$ and $\Psi$ is increasing, $\Psi(2\mathfrak{a}) \leq \Psi(d(\varpi, \mathcal{M}\varpi) < \Psi(2\mathfrak{a})$, a contraction. Hence $\mathcal{M}\varpi = \varpi, \varpi \in g(c_1, c_2, \mathfrak{a})$; that is, $g(c_1, c_2, \mathfrak{a})$ is a fixed ellipse of $\mathcal{M}$.

**Proof:** The proof obeys the method of Theorem 16. The subsequent instance elucidates Theorem 20 and 21.

**Example 2.4:** Let $\varXi =$
$\{-2, 0, (1/2)\ln(6/e), (1/2)\ \ln(15/e), (1/2), \ln(18/e), (1/2)\ln(21/e), (1/2)\ln(24/e), (1/2), \ln(27/e$
$(1/2), \ln(30/e), (1/2), \ln(6e), (1/2), \ln(9e), (1/2), \ln(12e), (1/2), \ln(15e), \ln 2, \ln 3, \ln 5\}$

and $d(\varpi, \mathfrak{d}) = |\varpi - \mathfrak{d}|\ \forall\ \varpi, \mathfrak{d} \in \varXi$ with $\varpi \perp \mathfrak{d} \iff \varpi \geq \mathfrak{d}$ be a OCMS. Let

$$\Psi(\mathrm{t}) = \mathrm{e}^t, c_1 = \ln 3\ , c_2 = \ln 5$$

$\mathfrak{a} = 1, \gamma = 0$ and $\alpha = 3/4$. Now,

$$g(\ln 3, \ln 5, 1) = \{\varpi \in \varXi : d(\ln 3, \varpi) + d(\ln 5, \varpi) = 1\}$$

$$= \{\varpi \in \varXi : |\ln 3 - \varpi| + |\ln 5 - \varpi| = 1\}$$

$$= \left\{\frac{1}{2}\ln\left(\frac{15}{e}\right), \frac{1}{2}\ln(15e)\right\}$$

$$\mathcal{M}\varpi = \begin{cases} 0\,, & \varpi = -2 \\ -2, & \varpi = 0 \\ \varpi \quad , & \text{otherwise} \end{cases}$$

then

$$d(\varpi, \mathcal{M}\varpi) = \begin{cases} 2, & \varpi \in \{-2,0\} \\ 0, & ,\text{otherwise} \end{cases}$$

Then $d(\varpi, \mathcal{M}\varpi) = 2 > 0$.

**Case 1**. For $\varpi = \{-2,0\}$ and $c_1 = \ln 3$

$$\left[\Psi\big(d(\varpi, \ln 3\,)\big)\right]^a \left[\Psi\big(d(\varpi, \mathcal{M}\varpi)\big)\right]^b \left[\Psi\big(d(\ln 3, \mathcal{M}\ln 3)\big)\right]^c \times \left[\Psi\big(d(\varpi, \mathcal{M}\ln 3)\big)\right]^e \left[\Psi\big(d(\ln 3, \mathcal{M}\varpi)\big)\right]^f$$

$$= \left[\Psi\big(d(\varpi - \ln 3\,)\big)\right]^a \left[\Psi\big(d(\varpi - \mathcal{M}\varpi)\big)\right]^b \left[\Psi\big(d(\ln 3 - \mathcal{M}\ln 3)\big)\right]^c \times \left[\Psi\big(d(\varpi - \mathcal{M}\ln 3)\big)\right]^e \left[\Psi\big(d(\ln 3 - \mathcal{M}\varpi)\big)\right]^f$$

$$= [\Psi(|\varpi - \ln 3|)]^a [\Psi(2)]^b [\Psi(|(\ln 3 - \mathcal{M}\ln 3)|)]^c [\Psi(|\varpi - \mathcal{M}\ln 3|)]^e \times [\Psi(|\ln 3 - \mathcal{M}\varpi|)]^f$$

$$= [\Psi(|\varpi - \ln 3|)]^{a+e} [\Psi(2)]^b [\Psi(|\ln 3 - \mathcal{M}\varpi|)]^f,$$

$$= [\Psi(|\varpi - ln3|)]^{a+e} [\Psi(2)]^b [\Psi(|\ln 3 - \mathcal{M}\varpi|)]^f,$$

$$= \begin{cases} [\Psi(\ln 3)]^{a+e} [\Psi(2)]^b [\Psi(|(\ln 3 + 2)|)]^f, \text{if } \varpi = 0 \\ [\Psi(|2 + \ln 3|)]^{a+e} [\Psi(2)]^b [\Psi(\ln 3)]^f, \text{if } \varpi = -2 \end{cases}$$

$$= \begin{cases} [\Psi(\ln 3)]^{a+e} [\Psi(2)]^b [\Psi(\ln\,(3\mathrm{e}^2))]^f, \text{if } \varpi = 0 \\ [\Psi(\ln\,(3\mathrm{e}^2)]^{a+e} [\Psi(2)]^b [\Psi(\ln 3)]^f, \text{if } \varpi = -2 \end{cases}$$

$$\begin{cases} 3^{a+e} e^{2b} (3e^2)^f, & \text{if } \varpi = 0 \\ (3e^2)^{a+e}\ e^{2b}\, 3^f \quad , & if\ \varpi = -2 \end{cases} > e^2 = \Psi(d(\varpi, \mathcal{M}\varpi)),$$

For $a = e = 1/4,\ b = 1/4, c = 0$, and $f = 1/3$, satisfying $a + b + c + e + f < 1$; that is,

$$\Psi\big(d(\varpi, \mathcal{M}\varpi)\big) < \left[\Psi\big(d(c_1, c_1)\big)\right]^a \left[\Psi\big(d(c_1, \mathcal{M}c_1)\big)\right]^b \left[\Psi\big(d(c_1, \mathcal{M}c_1)\big)\right]^c \left[\Psi\big(d(c_1, \mathcal{M}c_1)\big)\right]^e \left[\Psi\big(d(c_1, \mathcal{M}c_1)\big)\right]^f$$

**Case 2.** For $\varpi \in \{-2,0\}$ and $c_2 = \ln 5$,

$$\left[\Psi\big(d(\varpi, \ln 5\,)\big)\right]^a \left[\Psi\big(d(\varpi, \mathcal{M}\varpi)\big)\right]^b \left[\Psi\big(d(\ln 5, \mathcal{M}\ln 5)\big)\right]^c \times \left[\Psi\big(d(\varpi, \mathcal{M}\ln 5)\big)\right]^e \left[\Psi\big(d(\ln 5, \mathcal{M}\varpi)\big)\right]^f$$

$$= [\Psi(d(\varpi - \ln 5\,))]^{a}[\Psi(d(\varpi - \mathcal{M}\varpi))]^{b}[\Psi((\ln 5 - \mathcal{M}\ln 5))]^{c}$$
$$\times [\Psi(d(\varpi - \mathcal{M}\ln 5))]^{e}[\Psi((\ln 5 - \mathcal{M}\varpi))]^{f}$$

$$= [\Psi(|\varpi - \ln 5|)]^{a}[\Psi(2)]^{b}[\Psi(|(\ln 5 - \mathcal{M}\ln 5)|)]^{c} \times [\Psi(|\varpi - \mathcal{M}\ln 5|)]^{e}[\Psi(|\ln 5 - \mathcal{M}\varpi|)]^{f}$$

$$= [\Psi(|\varpi - \ln 5|)]^{a+e}[\Psi(2)]^{b}[\Psi(|\ln 5 - \mathcal{M}\varpi|)]^{f},$$

$$= [\Psi(|\varpi - \ln 5|)]^{a+e}[\Psi(2)]^{b}[\Psi(|\ln 5 - \mathcal{M}\varpi|)]^{f},$$

$$= \begin{cases} [\Psi(\ln 5)]^{a+e}[\Psi(2)]^{b}[\Psi(|(\ln 5 + 2)|)]^{f}, \text{if } \varpi = 0 \\ [\Psi(|2 + \ln 5|)]^{a+e}[\Psi(2)]^{b}[\Psi(\ln 5)]^{f}, \text{if } \varpi = -2 \end{cases}$$
$$= \begin{cases} [\Psi(\ln 5)]^{a+e}[\Psi(2)]^{b}[\Psi(\ln\,(5\mathrm{e}^{2}))]^{f}, \text{if } \varpi = 0 \\ [\Psi(\ln\,(5\mathrm{e}^{2})]^{a+e}[\Psi(2)]^{b}[\Psi(\ln 5)]^{f}, \text{if } \varpi = -2 \end{cases}$$

$$\begin{cases} 5^{a+e}e^{2b}(5e^{2})^{f}, \quad \text{if } \varpi = 0 \\ (5e^{2})^{a+e}\ e^{2b}\ 5\ ,\ \ if\ \varpi = -2 \end{cases} > e^{2} = \Psi(d(\varpi, \mathcal{M}\varpi)),$$

For $a = e = 1/4,\ b = 1/4, c = 0$, and $f = 1/3$, satisfying $a + b + c + e + f < 1$; that is,

$$\Psi(d(\varpi, \mathcal{M}\varpi)) < [\Psi(d(c_1, c_1))]^{a}[\Psi(d(c_1, \mathcal{M}c_1))]^{b}[\Psi(d(c_1, \mathcal{M}c_1))]^{c}[\Psi(d(c_1, \mathcal{M}c_1))]^{e}[\Psi(d(c_1, \mathcal{M}c_1))]^{f}$$

That is, $\mathcal{M}$ is an orthogonal Ψ-weak JS-contraction with $c_1 = \ln 3,\ c_2 = \ln 5$, and $a = e = 1/4,\ b = 1/4, c = 0$, and $f = 1/3$. Hence, $g(ln3, ln5,1) = \{(1/2)\ln(15/e), (1/2)\ln\,(15e)\}$ is a fixed ellipse and

$g_{\mathfrak{D}}(\ln 3, \ln 5,1) = \Xi\{-2,0\}$ be fixed elliptic disc of $\mathcal{M}$. Then $d(\ln 3, \varpi) + d(\ln 5, \varpi) \leq 1, \varpi \in g_{\mathfrak{D}}(\ln 3, \ln 5,1)$.

**Definition 2.5:** Let $\Theta: (0, \infty) \to (0, \infty)$ be an increasing function. A self-mapping $\mathcal{M}: \Xi \to \Xi$ O-preserving O-continuous of a OMS $(\Xi, \perp, d)$ is said to be a general integral-type orthogonal $\Theta_{\mathrm{g}}$-weak contraction with $\varpi \neq \mathfrak{d}$, if

$$d(\varpi, \mathcal{M}\varpi) > 0\ \Rightarrow \int_{0}^{\Theta \mathrm{d}(\mathrm{u}, \mathcal{M}\varpi)} \emptyset(t)dt \leq \int_{0}^{[\Theta(\mathfrak{L}(\mathrm{u}, \mathfrak{d}))]^{\alpha}} \emptyset(t)dt,$$

$$\mathfrak{L}(\mathrm{u}, \mathfrak{d}) = \max\{d(\varpi, \mathfrak{d}), \gamma d(\varpi, \mathcal{M}\varpi) + (1-\gamma)d(\mathfrak{d}, \mathcal{M}\mathfrak{d}), (1-\gamma)d(\varpi, \mathcal{M}\varpi) + \gamma d(\mathfrak{d}, \mathcal{M}\mathfrak{d}), \gamma d(\varpi, \mathcal{M}\mathfrak{d}) + (1-\gamma)d(\mathfrak{d}, \mathcal{M}\varpi), (1-\gamma)d(\varpi, \mathcal{M}\mathfrak{d}) + \gamma d(\mathfrak{d}, \mathcal{M}\varpi)\}, \gamma \in [0,1), \alpha \in (0,1), \varpi, \mathfrak{d} \in \Xi \text{ with } \varpi \perp \mathfrak{d}.$$

**Theorem 2.7:** Let $g(c_1, c_2, \mathfrak{a})$ be an ellipse in an OMS $(\Xi, \perp d)$ and $\mathfrak{a} = (1/2)\{\inf d(\varpi, \mathcal{M}\varpi): \varpi \neq \mathcal{M}\varpi\}$. If a self-mapping $\mathcal{M}: \Xi \to \Xi$be a O-preserving O-continuous and general integral-type orthogonal $\Theta_g$ -weak contraction $c_1, c_2 \in \Xi$ and $d(c_1, \mathcal{M}\varpi) + d(c_2, \mathcal{M}\varpi) = \mathfrak{a}, \varpi \in g(c_1, c_2, \mathfrak{a})$, then $g(c_1, c_2, \mathfrak{a})$ is a fixed ellipse of $\mathcal{M}$.

**Proof:** The proof obeys the method of Theorem 2.3

**Theorem 2.8:** If in Theorem 2.7 $d(c_1, \mathcal{M}\varpi) + d(c_2, \mathcal{M}\varpi) \leq \mathfrak{a}$, then is a fixed ellipse of $\mathcal{M}$.

Proof: The proof follows the pattern of Theorem 2.4

**Definition 2.6:** Let $\Psi: [0, \infty) \to [1, \infty)$ be an increasing function with $\Psi(0) = 1$; then map

$\mathcal{M}: \Xi \to \Xi$ of an OMS $(\Xi, d)$ is called an integral-type orthogonal $\Psi_g$-weak JS contraction with

$\varpi \neq \mathfrak{d}$, if $d(\varpi, \mathcal{M}\varpi) > 0$ implies that

$$\int_0^{\Psi(\, d(\varpi,\mathcal{M}\varpi))} \emptyset(t)dt$$

$$\leq \int_0^{[\Psi(\mathrm{d}(\varpi,\mathfrak{d}))]^a [\Psi(\mathrm{d}(\varpi,\mathcal{M}\varpi))]^b [\Theta(\mathrm{d}(\mathfrak{d},\mathcal{M}\mathfrak{d}))]^c [\Psi(\mathrm{d}(\varpi,\mathcal{M}\mathfrak{d}))]^e [\Psi(\mathrm{d}(\mathfrak{d},\mathcal{M}\varpi))]^f} \emptyset(t)dt$$

Where $a, b, c, e$ and $f$ are non-negative and $a + b + c + e + f \in [0,1), \varpi, \mathfrak{d} \in \Xi$ with $\perp, \mathfrak{d}$

**Theorem 2.9:** Let $g(c_1, c_2, \mathfrak{a})$ be an elliptic in an OMS $(\Xi, \perp, d)$ and $\mathfrak{a} = (1/2)\{inf d(\varpi, \mathcal{M}\varpi: \varpi \neq \mathcal{M}\varpi)\}$. If map $\mathcal{M}: \Xi \to \Xi$ be a O-preserving O-continuous integral-type orthogonal $\Psi_g$-weak JS contraction with $c_1, c_2 \in \Xi$ and $d(c_1, \mathcal{M}\varpi) + d(c_2, \mathcal{M}\varpi) = \mathfrak{a}, \varpi \in g(c_1, c_2, \mathfrak{a})$, then $g(c_1, c_2, \mathfrak{a})$ is fixed ellipse of $\mathcal{M}$.

Proof: The proof obeys the method of Theorem 2.3

**Theorem 2.10:** If in Theorem $26, d(c_1, \mathcal{M}\varpi) + d(c_2, \mathcal{M}\varpi) \leq \mathfrak{a}$, then $g(c_1, c_2, \mathfrak{a})$ is a fixed elliptic disc of $\mathcal{M}$.

**Proof:** The proof obeys the method of Theorem 2.4

## 3. Application

The concentration $\varpi(t)$ of the substance at time $t$ is given by:

$$-\frac{d^2\varpi}{dt^2} + \zeta(t)\varpi = \xi(t), \varpi(0) = \gamma, \varpi(1) = \delta \quad (3.1)$$

The function with initial value problem (IVP) (3.1) is

$$\varsigma(t,s)=\begin{cases} t(1-s), & 0\le t\le s\le 1,\\ s(1-t) & ,0\le t\le s\le 1\end{cases}$$

$$\varpi(t)=\gamma+(\delta-\gamma)t+\int_0^1 \varsigma(t,s)\left(\xi(s)-\zeta(s)\varpi(s)\right)ds, s\in[0,1]$$

Let $\Xi$ be a set of Riemann integrable functions on [0,1]; that is $\Xi=R[0,1]$ defined $d:\Xi\times\Xi\to\mathbb{R}^*$ by

$$d(\varpi,\mathfrak{d})=\|\varpi-\mathfrak{d}\|_\infty, \varpi,\mathfrak{d}\in\Xi, \qquad \text{with } \varpi\perp\mathfrak{d}$$

Where $\|\varpi\|_\infty=\sup_{t\in[0,1]}|\varpi(t)|$. Hence proved.

**Theorem 3.1:** Consider the boundary value problem (BVP) (48). Let $\mathcal{M}:\Xi\times\Xi\to\mathbb{R}$ be a self-map in a complete metric space $(\Xi,\perp,\mathrm{d})$, satistying

$$\|\varpi(t)-\mathfrak{d}(t)\|_\infty>0 \Rightarrow \|\zeta(t)\varpi(t)-\zeta(t)\mathfrak{d}(t)\|_\infty\le e^{-\lambda}\|\varpi(t)-\mathfrak{d}(t)\|_\infty, \lambda>0.$$

Proof: Define a map $\mathcal{M}:\Xi\to\Xi$ by

$$\mathcal{M}\varpi(t)=\gamma+(\delta-\gamma)t+\int_0^1 \varsigma(t,s)\left(\xi(s)-\zeta(s)\varpi(s)\right)ds, s\in[0,1]$$

Now, Since $\|\varpi(t)-\mathfrak{d}(t)\|_\infty>0$,

$$d(\mathcal{M}\varpi,\mathcal{M}\mathfrak{d})=\left|\gamma+(\delta-\gamma)t+\int_0^1 \varsigma(t,s)\left(\xi(s)-\zeta(s)\varpi(s)\right)ds-\gamma-(\delta-\gamma)t -\int_0^1 \varsigma(t,s)\left(\xi(s)-\zeta(s)\varpi(s)\right)ds\right|$$

$$=\left|\int_0^1 \varsigma(t,s)\left(\xi(s)-\zeta(s)\varpi(s)\right)ds-\int_0^1 \varsigma(t,s)\left(\xi(s)-\zeta(s)\varpi(s)\right)ds\right|$$

$$=\left|\int_0^1 \varsigma(t,s)\left(\zeta(s)\varpi(s)-\zeta(s)\mathfrak{d}(s)\right)ds\right|$$

$$<\|\zeta(s)\varpi(s)-\zeta(s)\mathfrak{d}(s)\|_\infty \sup_{t\in[0,1]}\left|\int_0^1\varsigma(t,s)\,ds\right|$$

$$<e^{-\lambda}\|\varpi(s)-\mathfrak{d}(s)\|_\infty \sup_{t\in[0,1]}\left|\int_0^1\varsigma(t,s)\,ds\right|$$

$$<\frac{1}{8}e^{-\lambda}\|\varpi(s)-\mathfrak{d}(s)\|_\infty=\frac{1}{8}e^{-\lambda}d(\varpi,\mathfrak{d})$$

If $\Theta(\mathrm{t})=\mathrm{e}^t, t\in(0,\infty)$, then

$$\Theta(\mathrm{d}(\mathcal{M}\varpi, \mathcal{M}\mathfrak{d}) = e^{d(\mathcal{M}\varpi, \mathcal{M}\mathfrak{d})} < e^{(1/8)e^{-\lambda}d(\varpi,\mathfrak{d})} = e^{d(\varpi,\mathfrak{d})^{((1/8)e^{-\lambda})}}$$

$$[\Theta(\mathrm{d}(\varpi, \mathfrak{d})]^{\alpha} \leq [\Theta(\max\{\mathrm{d}(\varpi, \mathfrak{d}), \mathrm{d}(\mathfrak{d}, \mathcal{M}\mathfrak{d}), d(\varpi, \mathcal{M}\varpi), d(\mathfrak{d}, \mathcal{M}\varpi), d(\varpi, \mathcal{M}\mathfrak{d})]^{\alpha}$$

Where $\alpha = (1/8)e^{-\lambda}$ and $\alpha \in (0,1)$. So, all the conditions of Theorem 8 are verified. Hence, $\mathcal{M}$ has a unique fixed point.

## 2. Conclusion

On an OCMS, we solve new directions as a fixed ellipse to the geometry of a set of non-unique fixed points. We arrange a special fixed point using an orthogonal Θ contraction and a Ciric-type orthogonal Θ contraction. Further research might be interesting in the context of a set of unique and non-unique fixed points.

**Data Availability:**

No data were used to support this study.

**Conflicts of Interest:**

The authors declare that they have no conflicts of interest.

## 3. References

[1] Al-Rawashdeh, A. and Ahmad, J.: Common fixed point theorems for JS-contractions, *Bulletin of Mathematical Analysis and Applications*, **8,** 12–22 (2016).

[2] Ahmad, J., Al-Mazrooei, A.E., Cho, Y. and Yang, Y.: Fixed point results for generalized Θ-contractions, *Journal of Nonlinear Sciences and Applications*, **10,** 2350–2358 (2017).

[3] Ahmad, A., Al-Rawashdeh, A.S., and Azam, A.: Fixed point results for {α, ξ}-expansive locally contractive mappings, *Journal of Inequalities and Applications*, 2014.

[4] Ahmad, J., Al-Rawashdeh, A., and Azam, A.: New fixed point theorems for generalized F-contractions in complete metric spaces, *Fixed Point Theory, and Applications*, 2015.

[5] Arshad, M., Ameer, E., and Karapinar, E.: Generalized contractions with triangular α-orbital admissible mapping on Branciari metric spaces, *Journal of Inequalities and Applications*, 63, 2016.

[6] Branciari, A.: A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, *Publicationes Mathematicae Debrecon*, **57,** 31-37 (2000).

[7] Banach, S.: Sur les op´erations dans les ensembles abstraits et leur application aux ´equations int´egrales, *Fundamenta Mathematicae*, **3,** 133–181 (1922).

[8] Baleanu, D., Rezapour, S., and Mohammadi, H.: Some existence results on nonlinear fractional differential equations, *Philosophical Transactions of the royal society*, 1–7 (2013).

[9] Caccioppoli, R.: Un teorema generale sull' esistenza di elementi uniti in una trasformazione funzionale, *Rendiconti Lincei*, **11,** 794–799 (1930).

[10] Ciric, L.J.B.: A generalization of Banach's contraction principle, *Proceeding of the American Mathematical Society*, **45,** 267–273 (1974).

[11] Das, P.: A fixed point theorem on a class of generalized metric spaces, *Korean Journal of Mathematics,* **9,** 29-33 (2002).

[12] Diminnie, C.R.: A new orthogonality relation for normed linear spaces, *Mathematische Nachrichten*, **114,** 197–203 (1983).

[13] Gordji, M.E., Rameani, M., de la Sen M., and Cho, Y.: On orthogonal sets and Banach fixed point theorem, *Fixed Point Theory*, **18,** 569–578 (2017).

[14] Jleli, M., and Samet, B.: A new generalization of the Banach contraction principle, *Journal of inequalities and applications*, 38, 2014.

[15] Jleli, M., Karapinar, E., and Samet, B.: Further generalization of the Banach contraction principle, *Journal of Inequalities and Applications*, 2014.

[16] Javed, K., Aydi, H., Uddin F., and Arshad, M.: On Orthogonal Partial b-Metric Spaces with an Application, Journal of Mathematics, Volume 2021, Article ID 6692063, 2021.

[17] Javed, K., Uddin, F., Aydi, H., Mukheimer, A., and Arshad, M.: Ordered-Theoretic Fixed Point Results in Fuzzy b-Metric Spaces with an Application, Journal of Mathematics Volume 2021, Article ID 6663707, 2021.

[18] Ramezani, M.: Orthogonal metric space, and convex contractions, *International Journal of Nonlinear Analysis and Applications*, **6,** 127-132 (2015).

[19] Samet, B., Vetro, C., and Vetro, P.: Fixed point theorems for α-ψ-contractive mappings, *Nonlinear Analysis,* **75**, 2154-2165 (2012).

[20] Suzuki, T.: Generalized metric spaces do not have the compatible topology, *Abstract, and Applied Analysis*, (2014).

[21] Sudsutad, W., and Tariboon, J.: Boundary value problems for fractional differential, *Advances in difference equations*, 93(2012).

[22] Uddin, F., Park, C., Javed, K., Arshad, M., and Lee, J.R.: Orthogonal m-metric spaces and an application to solve integral equations, *Advances in Difference Equations*, 2021.

[23] Uddin, F., Javed, K., Aydi, H., Ishtiaq, U., and Arshad, M.: Control Fuzzy Metric Spaces via Orthogonality with an Application, *Journal of Mathematics,* Volume 2021, Article ID 5551833, 2021.

[24] Aydi, H., Karapinar, E., & Yazidi, H., Modified $F$-Contractions via $\alpha$-Admissible Mappings and Application to Integral Equations. *Filomat*, **31,** 1141–1148 (2017).

[25] Aydi, H., Karapinar, E., Zhang, D., A note on generalized admissible-Meir-Keeler-contractions in the context of generalized metric space, Results in Mathematics, **71,** 73-92 (2017).

[26] Erdal Karapınar, Bessem Samet, "Generalized $\boldsymbol{\alpha}$-$\boldsymbol{\psi}$ Contractive Type Mappings and Related Fixed Point Theorems with Applications", *Abstract and Applied Analysis*, vol. 2012, Article ID 793486, 17 pages, 2012.

[27] Karapinar, E., A.F., fixed point on convex b-metric spaces via admissible mappings, TWMS journal of pure and applied mathematics, **12,** 254-264 (2021).

[28] Karapinar, E., Petruşel, A., & Petruşel, G. (2020). On admissible hybrid Geraghty contractions. *Carpathian Journal of Mathematics*, ***36*(3),** 433–442 (2020).

[29] Hussain, A., Ishtiaq, U., Ahmed, K. and Al-Sulami, H., 2022. On pentagonal controlled fuzzy metric spaces with an application to dynamic market equilibrium. *Journal of function spaces*, 2022.

[30] Uddin, F., Ishtiaq, U., Hussain, A., Javed, K., Al Sulami, H. and Ahmed, K., Neutrosophic Double Controlled Metric Spaces and Related Results with Application. *Fractal and Fractional*, **6(6)**, 318 (2022).

[31] Ali, U., Ishtiaq, U. and Ahmad, K., Statistically Convergent Sequences in Neutrosophic Metric Spaces. *Scientific Inquiry and Review,* ***6*(1)**, 2022**.**

[32] Uddin, F., Ishtiaq, U., Saleem, N., Ahmad, K. and Jarad, F., Fixed point theorems for controlled neutrosophic metric-like spaces. *AIMS Mathematics*, ***7*(12),** 20711-20739 (2022).

[33] Din, F.U., Javed, K., Ishtiaq, U., Ahmed, K., Arshad, M. and Park, C., Existence of fixed point results in neutrosophic metric-like spaces. *AIMS Mathematics*, ***7*(9)**, 17105-17122 (2022).

[34] Ahmad, K., Ishtiaq, U. and Afzal, J., An Application to Computer Science via New Fixed Point. Computer Science & Artificial Intelligence **2,** 1-13 (2022).

[35] M. Fréchet, Sur quelques points du calcul fonctionnel, palemo (30 via Ruggiero), 1906.

[36] M. Joshi, A. Tomar, and S. K. Padaliya, "Fixed point to fixed ellipse in metric spaces and discontinuous activation function," Applied Mathematics. E-Notes, **vol. 1,** 15 (2020).